The surface charges and the examples of J H Poynting on the transfer of energy in electrical circuits

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Abstract
In this paper we review the examples of J H Poynting (1884) on the transfer of electromagnetic energy in DC circuits. These examples were strongly criticized by O Heaviside (1887). Heaviside stated that Poynting had a misconception about the nature of the electric field in the vicinity of a wire through which a current flows. The historical review of this conflict and its resolution based on the consideration of electrical charges on the surface of the wires can be useful in the courses of electromagnetism or circuit theory.

Keywords: energy tranfer, surface charges, electric field, transmission line

1. Introduction

The space surrounding an electric circuit can be considered as a region where energy is transformed at certain points into electric and magnetic energy by means of batteries, generators, etc., while in other parts this electromagnetical energy is transformed into heat or mechanical work.

Poynting vector conceptualizes and quantifies the energy transport through the electromagnetic field and is generally used in undergraduate college courses as a way to represent the flow of energy of an electromagnetic wave. However, it may also be useful to represent the flow of energy in a DC circuit as in the examples presented by J.H. Poynting (JHP hereinafter) in 1884 [1], where he exposed the transfer of energy in electrical circuits by means of electromagnetic fields.

In his examples JHP assumed that the only component of the electric field was tangent to the surface of the resistive wire and, for that reason, the flow of energy that he calculated was perpendicular to the surface of the wires.

O Heaviside [2,3,4] (OH hereinafter) strongly criticized the results obtained by Poynting, he said that they was not correct and that, on the contrary, in real circuits the energy flow, parallel to the surface of the wires, is much greater than the energy flow in the normal direction.. OH states that this mistake is because JHP had a misconception about the nature of the electric field in the vicinity of a wire through which a current flows.

However, most authors and teachers, who discuss examples of energy transfer in a DC circuit, in their books, or in their classroom presentations, ignore Heaviside's criticism. In fact, in well-known physics texts it is exposed almost identically as JHP's explanation, as in the excellent book Lectures in Physics of R Feynman [5].

Since 1985, numerous authors, including Heald [6], Galili and Goibargh [7], Harbola [8] and Davis and Kaplan [9] noted that in order to achieve a good understanding of the transfer of electromagnetic energy in a circuit DC, it is necessary to know the electric field in the vicinity of the wires that carry a current. In particular, they showed that the surface charge on the wire creates two types of electric field: the field inside the wire that drives the current according to Ohm's law and the field outside the wire, which has components perpendicular and parallel to its surface. These authors' conclusions agree with the vision of Heaviside on the importance of the nature of the electric field surrounding a wire through which a current flows.

The plan of this work is the following: In Section 2 we examine the example of JHP on the energy flow in a straight wire carrying a current and also the criticism that this example has caused. In Section 3 we reproduce the qualitative reasoning OH and in Section 4, following this reasoning, we conduct a quantitative study, more contemporary, on the transfer of power in a transmission line. In Section 5 we examine the example of JHP on "discharge of a capacitor through a wire" which includes very arbitrary assumptions about the properties of the wire through which the discharge is performed. With these assumptions, we obtain an electric field tangent to the surface of the wire and an energy flow that penetrates in its interior in the normal direction, as in example 1. In Section 6, we solve the Laplace equation for the slow discharge of a capacitor through a cylindrical sheet (rather than a wire) for comparison with the example 2 from Poynting.

2. Example 1 of JHP: *A straight wire conveying a current.*

JHP presents this example as follows

"In this case very near the wire, and within it, the lines of magnetic force are circles around the axis of the wire. *The electric lines of force are along the wire*, if we take as proved that the flow across equal areas of the cross section is the same at all parts of the section.

If AB, Fig. 1, represents the wire, and the current is from A to B, then a tangent plane to the surface at any point contains the directions of both the electromotive and magnetic intensities (we shall write E.M.I. and M.I. respectively in what follows), and energy is therefore flowing in perpendicularly through the surface, that is, along the radius towards the axis. Let us take a portion of the wire bounded by two plane sections perpendicular to the axis.

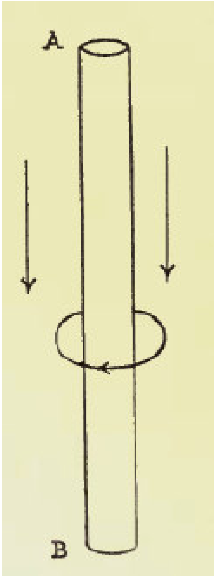

Figure 1. A wire carrying a current (Fig. 1 in Ref. [1])

Across the ends no energy is flowing, for they contain no component of the E.M.I. The whole of the energy then enters in through the external surface of the wire, and by the general theorem, the amount entering must just account for the heat developed owing to the resistance, since if the current is steady there is no other alteration of energy. It is, perhaps, worthwhile to show it independently in this case *where the energy moving inwards, in accordance with the general law, will just account for the heat developed*." (emphasis added).

However, OH [2-4], contemporary to JHP, who discovered the flux of energy from the electromagnetic field independently [10], pointed out that only a small component of this flux is directed perpendicularly to the wire but the other, much larger, is parallel to the wire. In his words [2], written in 1887:

"...the flow of energy takes place, in the vicinity of the wire, very nearly parallel to it, with a slight slope towards the wire. Prof. Poynting, on the other hand [Philosophical Transactions of the Royal Society, 1884] holds a different view, representing the transfer as nearly perpendicular to a wire, i.e., with a slight departure from the vertical. This difference of a quadrant can, I think, *only arise from what seems to be a misconception on his part as to the nature of the electric field in the vicinity of a wire supporting electric current*. The lines of electric force are nearly perpendicular to the wire. The departure from perpendicularity is usually so small that I have sometimes spoken of them as being perpendicular to it, as they practically are, before I recognized the great physical importance of the slight departure. It causes the convergence of energy into the wire. " (emphasis added).

The Example 1 of JHP. is used almost verbatim in the *Lectures on Physics* of R Feynman [5]. Indeed, in the Feynman Lectures we read: ‘‘We ask what happens in a piece of resistance wire when it is carrying a current. Since the wire has resistance, there is an electric field along it, driving the current. Because there is a potential drop along the wire, there is also an electric field just outside the wire, *parallel to the surface* (Fig. 27-5). There is, in addition, a magnetic field which goes around the wire because of the current. The $\boldsymbol{E}$ and $\boldsymbol{H}$ are at right angles; therefore there is a Poynting vector ($\boldsymbol{S} = \boldsymbol{E}\text{x}\boldsymbol{H}$) directed radially inward ... . There is a *flow of energy into the wire all around*. It is of course, equal to the energy being lost in the wire in the form of heat.” (emphasis added).

JHP does not hypothesize about the origin of the electric field inside the wire, however Feynman, argues that "... the electrons are really being pushed by an electric field, which has come *from some charges very far away*, and that the electrons get their energy to generate heat from these fields .... " (emphasis added). Feynman does not say it explicitly but it seems that places these "distant charges" on the terminals of the battery that powers the circuit.

Ohm's law is so simple that it is taken by many as almost self-evident. There is, however, a difficulty connected with the subject we discussed in this paper. This difficulty can be expressed in this question: What is the origin of the electric field $\boldsymbol{E}$ that directs the current inside the wire? Electric fields are produced by charges, so where are these charges? Consider, for instance, what happens when a resistance wire is connected to the plates of a battery. Before the connection is made there is a considerable electric charge on the plates, and hence a large electric force in the vicinity of the plates. After the connection has been made and transient currents have disappeared, there is a uniform electric field, $E = \rho J$, everywhere along the wire. Thus the electric force close to the plates has been reduced, while that at distant points has been increased. How has this come about? Galili [7] notes that this question that has also puzzled Feynman "can be clarified with a more suitable model for the electric current flowing in a wire. It is easy to understand that a constant current in a wire *implies a surface charge on the wire surface, guiding and pushing electrons*" (emphasis added).

Note that Galili (in 2004), as Heaviside (in 1887), points out that to understand the transfer of energy in a DC circuit requires a better understanding of the electrical fields surrounding the wire through which a current flows.

Model calculations of the surface charge for an infinite wire and for conductors of other geometries carrying direct current as well as RC circuits have been done [11,12]. The results of these investigations demonstrate that when there is current in the wire, both components of the electric field exist. Inside the wire there is only an axial component, but outside there is a perpendicular component as well. The electric field parallel to the surface gives rise to the energy flux that penetrates inside the wire. The perpendicular electric field component causes a

Poynting vector parallel to the wire. According to Heaviside this component is much larger than the other.

In the following session we are going to give Heaviside's qualitative explanation for the flow of energy in an electric circuit.

3. **The Heaviside approach to determining the flow of energy in a simple circuit**.

In [3], OH presents a qualitative, but very rigorous study of the flow of energy in an energy transmission line. His reasoning is as follows.

"In the vicinity of the wire the radial electric force varies inversely as the distance, and so does the intensity of magnetic force. The density of the energy-current therefore varies inversely as the square of the distance approximately. As regards the total energy-current, this is *VC*, the product of the fall of potential from one wire to the other into the current (*C*) in each. One factor, *V*, is the line-integral of the electric force across the dielectric. The other, *C*, is the line-integral ($1 \div 4\pi$) of the magnetic force round either wire."

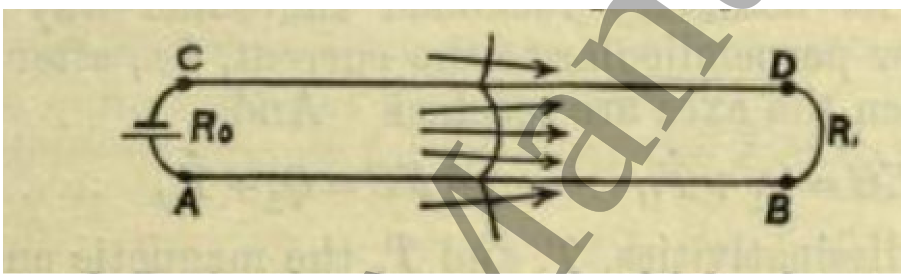

Figure2. A transmission line (from Ref. [3])

"In the figure (Fig. 2 in this work), *AB* and *CD* are the two wires, enormously shortened in length compared with their distance apart, joined through terminal resistances R_0 and R_1, in the former of which alone is the impressed force *e*. The fall of potential from *A* to *C* is V_0 from *B* to *D* is V_1, and at any intermediate distance is *V*. The total activity of the source is *eC*, of which $(e-V_0)C$ is wasted in R_0. What is left, or V_0C is the energy-current at *AC*, entering the line. By regular waste into the wires, its strength falls to V_1C at *BD*, where the line is left, and the terminal arrangement entered, to be wasted in frictional heat - generation R_1C^2 therein, or otherwise disposed of. The lines curved and arrows perpendicular to them show lines of electrical force and the direction of the energy-flow at a certain place, the inclination of the line of force to the perpendicular being greatly exaggerated, as well as that of the lines of flux of energy to the horizontal, in order to show the convergence of energy upon the wires, there to be wasted."

In the same text the magnitudes of the normal and tangential components of the electric field are compared:

"...we may compare the normal and tangential components of electrical force. Let there be a steady current in the straight wire, and the fall of potential from beginning to end be $V_0 - V_1$; the tangential component is then $(V_0 - V_1) \div l$. On the other hand, the fall of potential from the wire to its return – of no resistance, for simplicity – at any distance from the beginning of the line, is V, which is V_0 at one end and V_1 at the other. It is clear al once that the tangential is a exceedingly small fraction of the normal component of electric force, if the wire be long, and that it is only under quite exceptional circumstances anything but a small fraction. Prof. Poynting should therefore, I think, make his tubes of displacement stick nearly straight up as they travel along the wire, instead of having them nearly horizontal, unless I have greatly misunderstood him."

We can summarize the main arguments of OH:

1. The electric field inside the wire is determined by the potential drop, from the beginning to the end of the transmission line.

2. The V potential difference from one wire to the other (the return) is proportional to the electric field perpendicular to the wires.

3. The fall of potential from the wire to its return and, therefore, the normal electric field to the cables, decreases as we approach the end of the line.

OH does not explicitly mention the surface charges on the wires, but since the charges are proportional to the normal component of the electric field, they must decrease as we approach to the end of the line. This variation (gradient) of the surface charge density is responsible for the electric field inside the wire. If the wire is a perfect conductor, the tangential component of the electric field is zero and the surface charge density has a zero gradient.

The existence of the perpendicular component of the electric field is a consequence of the potential difference between the two wires of the transmission line, in other words, it is the result of the interaction between different parts of the circuit. In example 1, JHP analyzes a piece of wire that does not interact with the rest of the circuit and this leads him to ignore the normal component of the field. This neglect of JHP is not an obstacle to the correct calculation of the tangential component of the electric field and the flow of energy entering the interior of the wire. This partial achievement of his calculations possibly explains why this example has been repeated, in the same way, for over a hundred years in most texts that analyze the flow of electromagnetic energy in a circuit.

OH besides having a solid knowledge of Maxwell's equations, was an experimental electrician and this naturally led him to assert that in a real circuit the normal component of the electric field not only exists but is much larger than the tangential component. For OH, in a transmission line the energy must flow with the lowest attenuation possible.

In the next section we will make a quantitative study based on Heaviside qualitative reasoning.

4. Heaviside's approach with an up-to-date look

Heaviside's explanation of the previous section is linked to the new currents of investigation that try to explain the flow of energy with a model that takes into account the distribution of surface charges in the wires of the circuit. To confirm this link we will develop a more quantitative treatment of the Heaviside analysis.

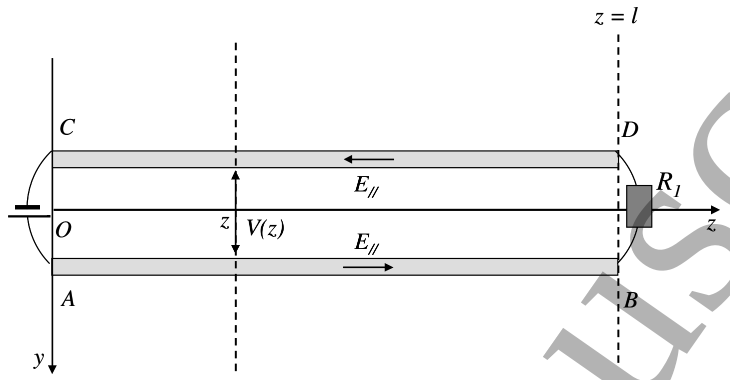

Figure 3. Two-wire transmission line, with radius *a* and separated by *d*.

A transmission line, consisting of two cylindrical wires of length l and radius *a,* is shown in Fig. 3. The wires are parallel to the z axis and are located in $(x=0, y=d/2)$ and $(x=0, y=-d/2)$. The battery is in $z=0$ and in $z=l$ the line is closed by a resistance R_1. To simplify, we assume that the internal resistance of the battery R_0 is zero. We also assume that $l >> d >> a$. The potential difference between the two wires is $V_0 = e$, at $z=0$, where *e* is the *emf* of the battery, and V_1 at $z=l$.

At any point z the potential difference between the wires is $V(z)$. If $E_{//}$ is the electric field inside the wires, then

$$V(z) = 2E_{//}(l-z) + V_1, \qquad (1)$$

The electric field $E_{//}$ and the potential drop between the cable ends are related by the following expression:

$$E_{//} = \frac{V_0 - V_1}{2l} \qquad (2)$$

Replacing (2) in (1) we obtain

$$V(z) = \frac{(V_0 - V_1)}{l}(l-z) + V_1 \qquad (3)$$

If $\Phi_{AB}(z)$ and $\Phi_{CD}(z)$ are the potentials of the *AB* and *CD* wires, at the z-point, then

$$V(z) = \Phi_{AB}(z) - \Phi_{CD}(z) = \int_{-d/2+a}^{d/2-a} E_y(y,z)dy \qquad (4)$$

Note that $\Phi_{AB}(z)$ and $\Phi_{CD}(z)$ are the values that the potential function takes at all points of the circumference of the conductors *AB* and *CD*.

In this potential difference is implicit the existence of an electric field that goes from the wire AB to the CD wire. This means that there is a component of the electric field perpendicular to the surface of the wires. This normal component of the electric field has been ignored by JHP and Feynman.

The charge on the surface of the wires is proportional to the normal component of the electric field, therefore, it must decrease with *z*, in the same way that decreases $V(z)$.

If $\lambda(z)$ is the electric charge per unit length on the wire *AB*, then:

$$\lambda(z) = cV(z), \tag{5}$$

where *c* is the capacitance per unit length, which can be interpreted as a "distributed capacity". The capacity per unit length of the transmission line is

$$c = \frac{\pi\varepsilon_0}{\ln\left[d/2a + \sqrt{(d/2a)^2 - 1}\right]}$$

As $d >> a$, then

$$c \approx \frac{\pi\varepsilon_0}{\ln(d/a)}$$

and

$$\lambda(z) = \frac{\pi\varepsilon_o}{\ln(d/a)}\left[\frac{(V_0 - V_1)}{l}(l - z) + V_1\right] \tag{6}$$

As the wires are separated by a distance that is much larger than its radius, the potential created by each wire agrees with the potential of a loaded line passing through the axis of the wires. Then, the potential of our two wires is obtained by superposing the potential of the two lines:

$$\Phi = \frac{-\lambda}{2\pi\varepsilon_0}\ln r_1 + \frac{\lambda}{2\pi\varepsilon_0}\ln r_2 = \frac{-\lambda}{2\pi\varepsilon_0}\ln\frac{r_1}{r_2} \tag{7}$$

Where r_1 and r_2 are the distances from the field point *P* to the axis of the wires *AB* and *CD*, respectively. References [13,14,15] calculate the potential and field created by a pair of resistive wires with equal and opposite currents, without the hypothesis ($d >> a$) we do.

The electric field components are calculated by taking the gradient of (7). In the *xy* plane

$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0}\frac{\vec{r}_1}{r_1^2} - \frac{\lambda}{2\pi\varepsilon_0}\frac{\vec{r}_2}{r_2^2} \tag{8}$$

Since λ depends on *z*, there is a component of the electric field parallel to the *z*-axis:

$$E_z = \frac{d\lambda/dz}{2\pi\varepsilon_0}\ln\frac{r_1}{r_2} \tag{9}$$

where

$$\frac{d\lambda}{dz} = -\frac{\pi\varepsilon_o}{\ln(d/a)}\frac{(V_0 - V_1)}{l} \tag{10}$$

The gradient in the density of the surface charge (10) provides the axial electric field within the wire that guides the movement of the conduction electrons. The normal component changes along the wire, reflecting the gradient of the surface charge.

On the surface of the wire *AB* the tangential component of the field is

$$E_z(y = d/2 - a, z) = \frac{V_0 - V_1}{2l} = E_{//} \tag{11}$$

If ρ is the resistivity of the wire and *S* its section, we can write $E_{//} = \rho I/S = \rho J$, where *J* is the current density. For continuity, this is the value that takes the electric field inside the wires.

The normal component (in the *yz* plane) of the electric field in the wire *AB* is

$$E_y(y = d/2 - a, z) = \frac{1}{2a\ln(d/a)}\left[\frac{(V_0 - V_1)}{l}(l - z) + V_1\right] \tag{12}$$

In the middle of the line, $z = l/2$,

$$E_y(y = d/2 - a, z = l/2) = \frac{(V_0 + V_1)}{4a\ln(d/a)} \tag{13}$$

In a "normal" transmission line, $l >> a$ y $V_0 - V_1 << V_0 + V_1$, then

$$E_y(y = d/2 - a, z) >> E_z(y = d/2 - a, z) \tag{14}$$

This confirms OH`s assertion: "...in an actual circuit the normal component of the electric field is much larger than the tangential component."

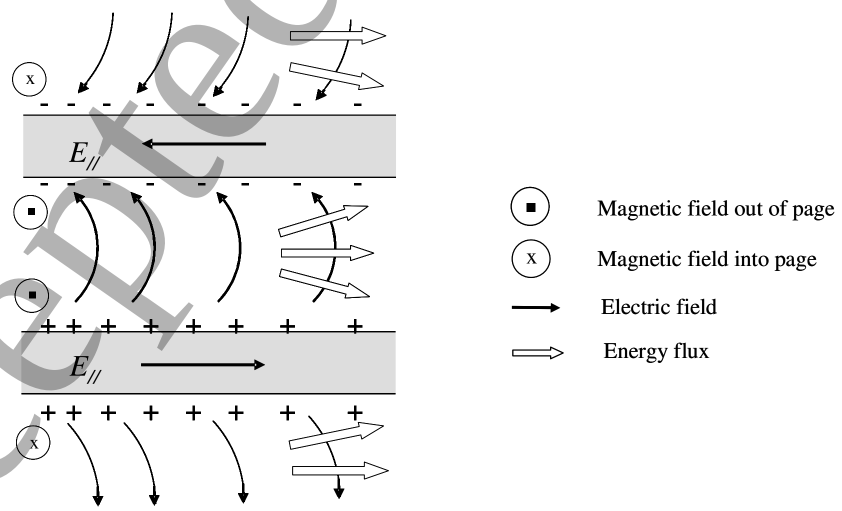

Figure 4. Surface charges, electric field, magnetic field and energy flow in a transmission line.

Like $d \gg a$, the magnetic field created by the wires is the sum of the magnetic field created by two filiform currents that coincide with the axes of the wires:

$$\vec{H} = \frac{I}{2\pi r_1}\frac{\hat{k}\times\vec{r}_1}{r_1^2} - \frac{I}{2\pi r_1}\frac{\hat{k}\times\vec{r}_2}{r_2^2} \qquad (15)$$

With the help of the previous results we can make the scheme of Fig. 4, where it is observed that the energy flow always from the battery to the resistive elements where the energy dissipates.

The magnetic field modulus which goes around the wire AB is approximately equal to $H \approx I/2\pi a$, since $d >> a$. This magnetic field and the electric field give rise to a flux of energy, normal to the surface of the wires, equal to $\rho I^2/2\pi a S$. The energy entering a length of wire of length Δz is $(\rho\Delta z/S)I^2$. The term in brackets is the resistance of this section of wire. Of course, it is equal to the energy that is lost in this portion of wire in the form of heat.

In order to calculate the electromagnetic power that is directed towards the terminal resistance of the line, we must calculate the flux of the Poynting vector through a surface that crosses the wires at the z-point.

$$\int_\Sigma (\vec{E}\times\vec{H}).\hat{k}da \qquad (16)$$

In (16) the versor $\vec{k}$ is parallel to the z-axis and Σ is the surface enclosed by the contour C shown in Fig. 5 . The calculation of this integral is very complicated; however, it can be done quickly if we use some integral theorems [16]. If we write $\vec{E}$ in terms of the electric potential Φ we get

$$\int_\Sigma (\vec{E}\times\vec{H}).\hat{k}da = -\int_\Sigma (\nabla\Phi\times\vec{H}).\hat{k}da \qquad (17)$$

A vector identity allows us to write this last integral of the form:

$$-\int_\Sigma (\nabla\Phi\times\vec{H}).\hat{k}da = -\int_\Sigma \nabla\times(\Phi\vec{H}).\,\hat{k}da + \int_\Sigma \Phi\nabla\times\vec{H}.\hat{k}da \qquad (18)$$

At the surface Σ the current is zero, then

$$-\int_\Sigma (\nabla\Phi\times\vec{H}).\hat{k}da = -\int_\Sigma \nabla\times(\Phi\vec{H}).\,\hat{k}da \qquad (19)$$

Using Stokes' law the first integral can be converted into a line integral

$$\int_\Sigma (\vec{E}\times\vec{H}).\hat{k}\,da = -\oint_C \Phi\vec{H}.d\vec{s} \qquad (20)$$

The contributions to this integral vanish at the infinite part of the contour C. In fact, when r becomes very large, the integrand $\Phi\vec{H}$ tends to zero as $1/r^3$, and the length of contour C grows as r, then the integral tends to zero as $\left(1/r^3\right)r = 1/r^2$. The contributions along the segments connecting C_1 and C_2 to infinity cancel, and so the only contribution comes from C_1 and C_2. According to Ampere's law, the circulation along C_1 is $-I$ and the circulation along C_2 is I, then

$$\int_\Sigma (\vec{E}\times\vec{H}).\hat{k}\,da = \Phi_{AB}(z)I - \Phi_{CD}(z)I = V(z)I \tag{21}$$

We see that the integral of the Poynting flow over the cross-section of the system give us simply $V(z)I$.

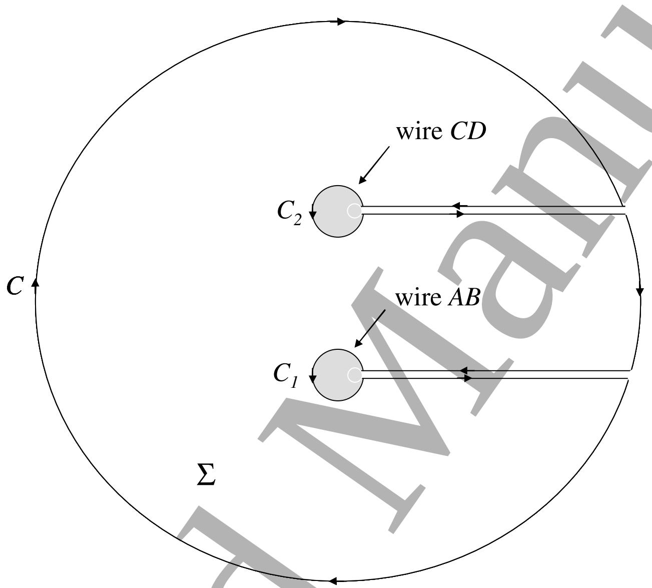

Figure 5. Cross-section of the transmission lines. Area Σ is enclosed by the curves $C + C_1 + C_2$

If we replace (3) in (21) we obtain

$$\int_\Sigma (\vec{E}\times\vec{H}).\hat{k}\,da = 2\left[\rho(l-z)/S\right]I^2 + V_1 I \tag{22}$$

The first term is the power dissipated in the AB and CD wires in the span from z to l ($\rho(l-z)/S$ the is the resistance of one of the wires of length $l-z$) and the second term $V_1 I$ is the dissipated power at the terminal resistance R_1.

5. Example 2 of JHP: *Discharge of a condenser through a wire*.

In this example JHP investigates how the energy travels through the medium on its way to the resistive wire. His argument is as follows:

"We shall first consider the case of the slow discharge of a simple condenser consisting of two charged parallel plates when connected by a wire of very great resistance, as in this case we can form an approximate idea of the actual path of the energy.

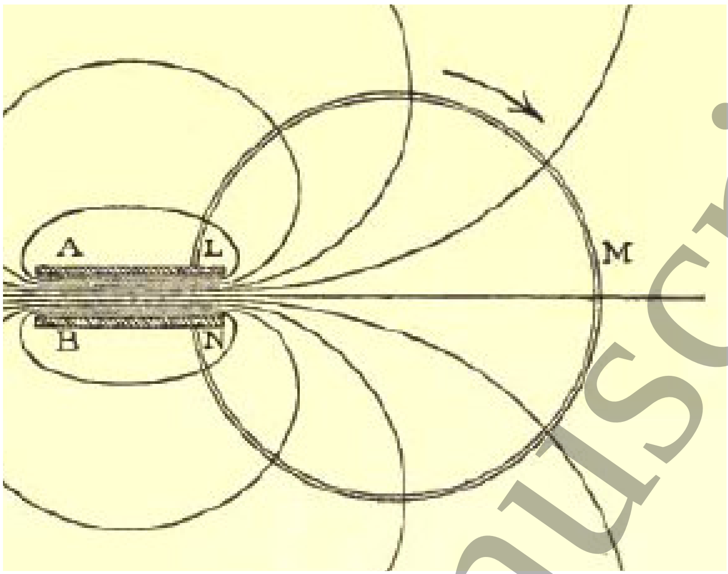

Figure 6. Discharge of a condenser through a wire (Fig 2 in [1])

Let A and B, Fig. 2 (Fig. 6 in this work), be the two plates of the condenser, A being positively and B negatively electrified. Then before discharge the sections of the equipotential surfaces will be somewhat as sketched. The chief part of the energy resides in the part of the dielectric between the two plates, but there will be some energy wherever there is electromotive intensity. Between A and B the E.M.I. will be from A to B, and everywhere it is perpendicular to the level surfaces. Now connect A and B by a fine wire LMN of very great resistance, following a line of force and with the resistance so adjusted that it is the same for the same fall of potential throughout. We have supposed this arrangement of the resistance so that the level surfaces shall not be disturbed by the flow of the current. The wire is to be supposed so fine that the discharge takes place very slowly.

While the discharge goes on a current flows round LMN in the direction indicated by the arrow, and there is also an equal displacement-current from B to A due to the yielding of the displacement there. The current will be encircled by lines of magnetic force, which will in general form closed curves embracing the circuit. The direction of this around the wire will be from right to left in front, and around the space between A and B from left to right in front. The E.M.I. is always from the higher level surfaces — those nearer A, to the lower — those nearer B, both near the wire and in the space between A and B.

Now, since the energy always moves perpendicularly to the hues of E.M.I. it must travel along the equipotential surfaces. Since it also moves perpendicularly to the lines of M.I. it moves, as we have seen in case No. (1), inwards on all sides to the wire, and it is all converted into heat — if we suppose the discharge so slow that the current is steady during the time considered. But between A and B the E.M.I. is opposed to the current, being downwards, while the M.I. bears the same relation to the current as in the wire. Remembering that E.M.I., M.I., and the direction of flow of energy are connected by the right-handed screw relation, we see that the energy moves outwards from the space between A and B. As then the strain of the dielectric between A and B is gradually released by what we call a discharge current along the wire LMN, the energy thus given up travels outwards through the dielectric, following always the equipotential surfaces, and gradually converges once more on the circuit where the surfaces are cut by the wire. There the energy is transformed into heat. It is to be noticed that if the current may be considered steady, the energy moves along at the same level throughout."

This example of JHP offers several interesting aspects to be analyzed. First, he chooses the shape of the LMN wire to match a line of electric field force created by the electrical charges that are on the outer surfaces of the capacitor. This electric field then directs the current inside the wire. Since electric current I must have the same value at all points of this wire, JHP accommodates its resistivity so that Ohm's law is fulfilled in each segment. This implies that

$$\rho = \frac{ES}{I} \tag{23}$$

The electric field E is relatively strong in the vicinity of the plates, near the points L and N, and decreases rapidly when we move towards the point M. The resistivity of the wire, according to (23), must vary in the same way. There is no doubt that this cable is very special.

What happens if the *LMN* cable (wire) is a ordinary cable with a constant conductivity? In this situation the law of Ohm determines that the electric field must have a constant modulus along the whole cable. But, what is the origin of the electric field in the places of the wire that are far from the plates of the capacitor? Clearly the E field can be created only by electric charges. The electric charges on the capacitor plates can act appreciably only in their near areas, since the electric field decreases as the square of the distance. Then, the charges on the capacitor plates are not the ones that create this electric field of constant modulus inside the wire.

We now know that the charges that create this electric field are charges that are distributed over the surface of the wires. JHP and R Feynman have not been aware of the role of surface charges in wires in creating the electric field that directs the current inside them. Both assume that the charges that create this field are very far away, Feynman seems to suggest that they are over the battery terminals while JHP places them on the outer surface of the capacitor plates. The field at an inner point of the wire, however, is created by surface electrical charges that are close to that point. Hartel [17] states that the influence of the surface charge extends only to distances that are comparable with the diameter of the wire.

6. Capacitor's discharge through a cylindrical conductor

If we try to solve the Laplace equation for the electrostatic potential in the geometry of Fig. 7 (but, with a wire of constant resistivity) we would have to deal with the boundary conditions on the toroidal surface of a wire with finite radius. In principle, the problem can be solved using numerical methods, but the contour geometry and the three-dimensionality make it very difficult.

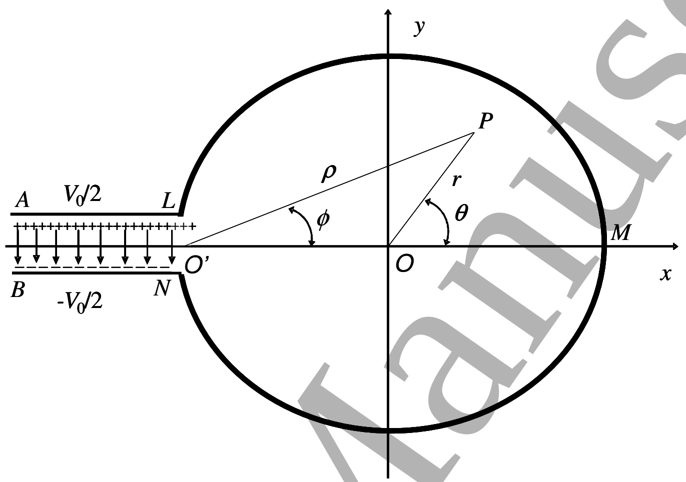

Figure 7. Discharge of a capacitor through a cylindrical resistive sheet. r,θ are the polar coordinates with center in O and ρ,ϕ are the polar coordinates with center in O'.

For these reasons, we restrict our attention to a two-dimensional analog problem outlined in Fig. 7. We will interpret Fig.7 as the cross-sectional representation of a device composed of a capacitor which is connected to a very long cylinder of radius a. The capacitor is charged until the potential difference between the plates is V_0. When the resistive cylindrical surface is connected to the capacitor plates, a current will start to flow. If the resistance of the cylinder is very great the discharge will be very slow and the current will remain relatively constant during a certain interval of time. With these conditions this problem is similar to those solved in [6,15].

The capacitor plates are located in $\theta \approx \pm\pi$ (if we assume $d << a$) and its plates are at potentials $\pm V_0 / 2$. We use a conventional cylindrical coordinate system r,θ,z coaxial with the cylinder as shown in Fig. 7. In this figure ρ,ϕ are the polar coordinates with center in O'.

In accordance with Ohm's law, the potential on the inner surface of the resistive cylinder is

$$\Phi(a,\theta) = \frac{V_0\theta}{2\pi} \qquad (-\pi < \theta < \pi) \tag{24}$$

The solution of the Laplace equation, inside the cylinder is given by (see Ref. [6])

$$\Phi(r < a;\theta) = \left(\frac{V_0}{\pi}\right)\arctan\left[r\,\mathrm{sen}\,\theta / (a + r\cos\theta)\right] = \left(\frac{V_0}{\pi}\right)\phi, \tag{25}$$

Inside the cylinder the equipotential are the planes $\phi = cte$. The electric field at any point is found by calculating the potential gradient. The components of the electric field inside the cylinder are:

$$E_\theta = -\frac{1}{r}\frac{\partial\Phi}{\partial\theta} = -\frac{V_0}{\pi}\frac{(r + a\cos\theta)}{\rho^2} \tag{26}$$

$$E_r = -\frac{\partial\Phi}{\partial r} = -\frac{V_0}{\pi}\frac{a\,\mathrm{sen}\,\theta}{\rho^2} \tag{27}$$

The electric field force lines, perpendicular to the equipotentials, are shown in Fig. 8. Note that the lines of force are circumferences centered in $x = -a,\ y = 0$.

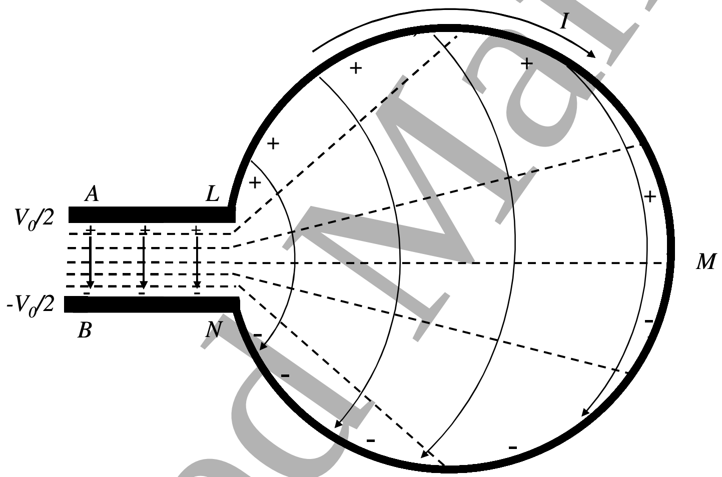

Figure 8. Equipotentials lines, champ electric and surface charges. The equipotentials lines inside the cylindrical resistive sheet can also be interpreted as lines of energy flux. In this figure, in order to observe the equipotential lines inside the capacitor we have magnified the distance d between the plates.

At the point $(r = a, \theta)$ of the cylinder surface, the tangential component is

$$E_\theta(r = a,\theta) = -\frac{V_0}{2\pi a} \tag{28}$$

and the normal component is

$$E_r(r = a,\theta) = -\frac{V_0}{2\pi a}\tan\left(\frac{\theta}{2}\right) = -\frac{V_0}{2\pi a}\tan\phi \tag{29}$$

The surface charge density on the inner surface of the cylinder is

$$\sigma(\theta) = -\varepsilon_0 E_r(r=a,\theta) = \frac{\varepsilon_0 V_0}{2\pi a}\tan\phi \qquad (30)$$

Note that $\sigma(\theta)$ does not vary linearly along the perimeter of the resistive cylinder as in the case of the two-wire transmission line, but is a tangent function that increases non-linearly near the battery.

The magnetic field inside the cylinder is $H_z = -I/w$, where w is the length of the cylinder. Poynting vector is in the xy plane and is orthogonal to the electric field lines. Then, the equipotential lines in Fig. 8 are also lines of the energy flow. The energy flow is directed along a straight line from the inside of the capacitor to each of the resistive elements of the cylinder.

The electric field between the plates of the capacitor is $E_y = -V_0/d$, then the Poynting vector, in $x=-a,\ y=0$, (at the right end of the capacitor) is $P_{cx} = VI/dw$. The total power flowing from the condenser to the surface of the resistive cylinder is $W_c = P_{cx}(dw) = VI$.

The normal component of the Poynting vector $P_r(r=a,\theta)$ on the surface of the cylinder is $VI/2\pi aw$ and the total power entering the cylinder is *VI*. In conclusion, the power dissipated in the resistive cylinder comes from the energy stored in the capacitor.

Although the calculations presented in this section are only for a two-dimensional (cylindrical) case, the results allow us to get an idea of the surface distributions of electric charges, equipotential surfaces, electric fields and the flow of energy in any electric circuit simple in three-dimensional space. In particular, this example shows how energy flows from the inside of the condenser to the resistive cylinder elements acting as sinks. This was the goal of Example 2 of JHP

7. Conclusion

The Poynting vector allows us to visualize the flow of energy from the battery to the place where it is consumed. This description needs the knowledge of the electric and magnetic fields that surround the circuit. OH asserted that JHP's explanation in his first example of the transfer of electromagnetic energy in 1885 was not correct because he had a misconception about the nature of the electric field around a wire through which a current flows. This misconception, however, lasted over time and is found in most textbooks dealing with this subject, for example, in reference [5].

For this reason, objections similar to those formulated by Heavisade have appeared in articles of numerous investigators since 1985. These researchers point out that in order to achieve an adequate understanding of the flow of electromagnetic energy it is necessary to know the surface charges in the circuit wires. However, the dissemination of the surface charge approach in learning materials is slow.

We believe that the diffusion we make of the controversy between Poynting and Heaviside, which is of the year 1885, can help to achieve a more correct treatment of this question from the vision based on the surface charges. The analysis of circuits in terms of surface charges give answers to some questions that do not have an appropriate explanation within the context of traditional circuit theory: Who creates the electric field that moves the charges inside the conductors? How does energy flow from the battery to the resistive elements where it dissipates? These questions cannot be answered within the framework of traditional circuit theory which is based on the concept of potential difference. The charge density gradient, on the surface of the resistive elements, creates the electric field that produces current flow. This electric field is the source of the potential difference along the resistive element. The surface charges that are required to create the electric field, which maintains the current, also produce, in the outer space of the conductors, the electric field that is needed for the transfer of energy.

Appendix

We consider it convenient to include in this appendix simple calculations showing the magnitude difference between the Heaviside and Poynting energy flux vectors and that despite this enormous difference the energy balance remains valid.

From the results of section 4 it follows that the tangential component of the electric field is given by $E_{//} = R_l I/l$ where R_l is the resistance of one of the wires of the line and I is the current in the circuit. The normal component of the electric field is given approximately by $E_n \approx R_1 I/\left(a \ln d/a\right)$, where R_1 is the resistance at the end of the line (see figure 3). Then,

$$\frac{S_{//}}{S_n} = \frac{E_n}{E_{//}} \approx \frac{R_1}{R_l}\frac{l}{2a \ln d/a},$$

where $S_{//}$ and S_n are the components of the energy flow in the parallel and normal direction to the wire, respectively,

If $R_1 = 98\,\Omega$, $R_l = 1\Omega$ and l = 1000 *m* (a telegraph line), d = 10 *cm* and a = 0.5 *cm*, then on the surface of the wires $S_{//}/S_n \approx 10^7$. Away from the transmission line $S_{//} \rightarrow 0$ like $1/r^3$ when $r \rightarrow \infty$

For Heaviside, a great theorist of electromagnetism, but also a practical electrician (he worked on the design and laying of transoceanic telegraph lines) the transmission line must have very little resistance so that the energy arrives practically without dissipating in the terminal resistance. This is the reason why he claimed that the parallel component of the energy flow, in a real circuit, is much larger than the normal component.

If in the previous example the voltage at $z = 0$ is $V_0 = 100V$, then the current in the circuit is $I = 1A$, the power dissipated in the line is 2 W while the terminal resistance dissipates 98 W. The power supplied by the battery is 100 W.

If we calculate, for this example, the flow of the component "Heaviside" on an infinite surface Σ that cuts the line at a point z that is to 500 *m* from the battery, we find (Eq. 22) that 1*W* dissipates in the section of the line that goes from 500 to 1000 *m* and 98*W* dissipate in the terminal resistance.

This is a trivial result of circuit theory. However, this theory does not tell us how energy travels from the battery to the different parts of the circuit where it dissipates. The theory of Poyting-Heaviside tells us that this energy leaves the battery, traveling throughout the space that surrounds the circuit, and converges on the different points of the circuit where it dissipates.

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