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Lie bialgebra structures on 2-step nilpotent graph algebras

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Abstract

We generalize a result on the Heisenberg Lie algebra that gives restrictions to possible Lie bialgebra cobrackets on 2-step nilpotent algebras with some additional properties. For the class of 2-step nilpotent Lie algebras coming from graphs, we describe these extra properties in a very easy graph-combinatorial way. We exhibit applications for $\mathfrak{f}_n$, the free 2-step nilpotent Lie algebra.

Introduction

Thorough this paper, all vector spaces are over a fixed field k of characteristic different from 2 and 3, so that $\Lambda^2\mathfrak{g}$ and $\Lambda^3\mathfrak{g}$ can be considered as subspaces of $\mathfrak{g}^{\otimes 2}$ and $\mathfrak{g}^{\otimes 3}$, respectively. A **Lie bialgebra** is a triple $(\mathfrak{g}, [-,-], \delta)$ where $(\mathfrak{g}, [-,-])$ is a Lie algebra and $\delta : \mathfrak{g} \to \Lambda^2\mathfrak{g}$ is a linear map such that

- δ satisfies the co - Jacobi identity. In Sweedler type notation, if $\delta(x) = x_1 \wedge x_2$ (sum understood), then co-Jacobi condition reads

$$\delta(x_1) \wedge x_2 - x_1 \wedge \delta(x_2) = 0 \in \Lambda^3\mathfrak{g}$$

- δ satisfies the 1-cocycle condition $\delta[x,y] = [\delta x, y] + [x, \delta y] \in \Lambda^2\mathfrak{g}$.

In this paper we deal with the classification problem when the underlying Lie algebra is 2-step nilpotent. Lie bialgebras first appear as classical limit of quantum objects, studying deformations of Hopf algebras and the Quantum Yang-Baxter equation, presented by Drinfel'd. They also appear in geometry as Poisson Lie structures on Lie

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group, and in algebraic topology as additional algebraic structure on the space of (free homotopy classes of) curves in oriented surfaces. Lie bialgebra structures with underlying complex simple Lie algebra were classified by Belavin and Drinfel'd [BD] under a non-degenerated assumption (called factorisable case) and after that many mathematicians have worked on semisimple and reductive cases. In [FJ] we studied the situation of a Lie algebra $\mathfrak{l} = \mathfrak{g} \times V$, where V is an abelian factor, and the problem of determine all possible Lie bialgebra structures supported by $\mathfrak{l}$ in terms of Lie bialgebra structures on $\mathfrak{g}$; we obtain in that way non factorisable examples for reductive Lie algebras. Although its importance, the classification problem of Lie bialgebras remain a wide open problem. In the opposite side of simple or semisimple case (e.g. solvable or nilpotent) there are almost no results, and in some cases is hopeless for obvious reasons: if the underlying algebra $\mathfrak{g}$ is abelian (say $\dim \mathfrak{g} < \infty$), to classify all Lie bialgebra structures on $\mathfrak{g}$ is the same as the classification of all Lie algebra structures on the vector space $\mathfrak{g}^*$. Surprisingly, if $\mathfrak{g} = \mathfrak{h}_{2m+1}$, the 2m+1-dimensional Heisenberg Lie algebra, all Lie bialgebra structures on $\mathfrak{h}_{2m+1}$ are known from a long time (see [BS] and [SZ]), but for general nilpotent Lie algebras there is nothing else. The nilpotent case is of special importance because after a result of Etingof and Gelaki (see [EG]), connected Hopf algebras of finite Gelfand - Kirillov dimension are in bijection with nilpotent Lie bialgebras.

The origin of this work is the following: if $\mathfrak{n}$ is a 2-step nilpotent Lie algebra with center $\mathfrak{z}$, then one can find a linear complement W so that $\mathfrak{n} = W \oplus \mathfrak{z}$. Once this choice is made, one can see analogies between our previous work ([FJ], $\mathfrak{l} = \mathfrak{g} \times V$) and the Heisenberg case $\mathfrak{h}_{2m+1} = W \oplus kz$. Using this point of view we obtain, under mild assumptions, very strong restrictions on all possible co-bracket on a general 2-step nilpotent algebra. Although these restrictions, some nonlinear equations remains to be solved and the general situation is still wide, but restricted to graph algebras the problem is more tractable. Graphs algebras is a very interesting class of 2-step nilpotent algebras (see definition at the begining of section 2), even if in each dimension there are only a finite number of them, they are numerous enough to construct interesting examples and counterexamples in geometry (see [LW] and [GGI]).

The results of this paper are:

In section 1 we begin with a simple but very useful fact (Corollary 1.3): if $(\Lambda^2\mathfrak{n})^{\mathfrak{n}} = \Lambda^2\mathfrak{z}$ then $\delta(\mathfrak{z}) \subseteq \Lambda^2\mathfrak{z}$ for any cobracket δ. The main result of section 1 is **Theorem 1.4**, where the Heisenberg case is generalized to arbitrary 2-step algebras: if $\delta(\mathfrak{z}) \subseteq \Lambda^2\mathfrak{z}$ and a given linear system of equations has unique solution then $\delta(W) \subseteq \mathfrak{n} \wedge \mathfrak{z}$, that is, $\delta(W)$ does not have $\Lambda^2 W$-component. We call TST type the class of 2-step algebras satisfying this linear condition (see definition 1.7 and notations introduced after Corollary 1.3). In **Theorem 1.9** we give the precise algebraic data determining a Lie bialgebra structure on a Lie algebra $\mathfrak{n}$ satisfying $\delta(\mathfrak{z}) \subseteq \Lambda^2\mathfrak{z}$ and $\delta(W) \subseteq \mathfrak{n} \wedge \mathfrak{z}$. **Theorem 1.10** is a reciproque for a TST type algebra satisfying $(\Lambda^2\mathfrak{n})^{\mathfrak{n}} = \Lambda^2\mathfrak{z}$.

In section 2 we specialize to graph algebras. We prove in **Theorem 2.4** that if $\mathfrak{n}$ is a 2-step nilpotent algebra coming from a graph G, then $(\Lambda^2\mathfrak{n})^{\mathfrak{n}} = \Lambda^2\mathfrak{z}$ if and only if the degree of each vertex is bigger or equal to 2. In **Theorem 2.6** we indicate conditions on the graph that imply TST type. In particular, if the graph is "highly connected " (e.g. the degree of each vertex is bigger or equal to 2) then its associated algebra easily satisfies both $(\Lambda^2\mathfrak{n})^{\mathfrak{n}} = \Lambda^2\mathfrak{z}$ and the TST condition.

Section 3 deals with an application of a special class of Lie bialgebras, the ones whose co-bracket annihilates the center (this class largely includes the coboundary ones). Inside this class we study the case $\mathfrak{n} = \mathfrak{f}_n$, the free 2-step nilpotent algebra on n generators, that in terms of graph corresponds to the complete graph K_n. The main fact is **Corollary 3.8**: If $n \geq 4$ and δ a bialgebra structure on $\mathfrak{f}_n$ satisfying $\delta(\mathfrak{z}) = 0$ and D_α diagonalizable for all α, then $\delta(W) \subset \Lambda^2\mathfrak{z}$, hence, $\mathfrak{n}$ is also 2-step conilpotent. (The notation for the D_α's are as in Theorem 1.9.) We also present computations in the non diagonalizable case for $n = 3$.

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1 Two-step nilpotent Lie bialgebras

A Lie algebra $\mathfrak{n}$ is called 2-step nilpotent if $[\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] = 0$, or equivalently if $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{z}$, where $\mathfrak{z}$ is the center of $\mathfrak{n}$. As vector space, the center admits a linear complement; we choose one and call it W, so that $\mathfrak{n} = W \oplus \mathfrak{z}$ as linear vector spaces. We will study Lie cobrackets δ defined on the underlying Lie algebra $\mathfrak{n}$. If $(\mathfrak{n}, \delta)$ is a Lie bialgebra structure then $\delta : \mathfrak{n} \to \Lambda^2\mathfrak{n}$, admits a decomposition via

$$\mathfrak{n} = W \oplus \mathfrak{z} \xrightarrow{\delta} \Lambda^2(W \oplus \mathfrak{z}) = \Lambda^2 W \oplus W \wedge \mathfrak{z} \oplus \Lambda^2\mathfrak{z}$$

So, we may split δ in several components. In particular, define $\delta_1 := \delta|_W^{\Lambda^2 W}$, that is, for $v \in W$, $\delta_1(v)$ is the $\Lambda^2 W$ - component of $\delta(v)$.

A very important and classical example is the Heisenberg Lie algebra $\mathfrak{n} = \mathfrak{h}_{2n+1} = W \oplus kz$ where $W = k^{2n}$ with symplectic form ω and bracket given by

$$[v + \lambda z, w + \mu z] = \omega(v, w)z$$

For the Heisenberg Lie algebra, all possible cobrackets are well-known

Theorem 1.1. *([SZ] and [BS]) If $\delta(z) = v_0 \wedge z$ then δ is of the form*

$$\begin{array}{rcll} \delta(v) & = & \frac{1}{4}(\omega(v_0, v)\widehat{\omega} + v_0 \wedge v) & \in \Lambda^2 W \\ & & +D(v) \wedge z & \in W \wedge z \end{array}$$

where D is a coderivation with respect to δ_1, and $v \in W$.

In particular, δ_1 **is determined** by $\delta(z)$, and if $\delta(\mathfrak{z}) = 0$ then $\delta(v) = D(v) \wedge z\ \forall v \in W$. We will generalize one direction of the above theorem. We begin with a simple remark

Remark 1.2. If $\delta : \mathfrak{n} \to \Lambda^2\mathfrak{n}$ is a 1-cocycle then $\delta(\mathfrak{z}) \subset (\Lambda^2\mathfrak{n})^{\mathfrak{n}}$.

Proof. For $x \in \mathfrak{n}, z \in \mathfrak{z}$, $\mathrm{ad}_z \delta x = [z, x_1] \wedge x_2 + x_1 \wedge [z, x_2] = 0$, hence

$$0 = \delta(0) = \delta[x, z] = \mathrm{ad}_x \delta z - \mathrm{ad}_z \delta x = \mathrm{ad}_x \delta z + 0$$

That is, δz is ad_x-invariant for any $x \in \mathfrak{n}, z \in \mathfrak{z}$. □

Corollary 1.3. *If* $(\Lambda^2 \mathfrak{n})^{\mathfrak{n}} = \Lambda^2 \mathfrak{z}$ *then* $\delta(\mathfrak{z}) \subset \Lambda^2 \mathfrak{z}$.

Even if the Heisenberg Lie algebra does not satisfy $(\Lambda^2 \mathfrak{n})^{\mathfrak{n}} = \Lambda^2 \mathfrak{z}$, several interesting 2-step nilpotent Lie algebras do, as we will show in next section. Now the important simplification will be given by next theorem. We need to introduce some notation.

For $v, w \in W$, $[v, w] \in \mathfrak{z}$, fix a basis $\{z_i\}_{i=1}^{\dim \mathfrak{z}}$, and write

$$[v, w] = \sum_i T_i(v)(w) z_i$$

where $T_i : W \to W^*$ is determined by the Lie bracket.

Denote by $[\ ,\ ]^* : W^* \times W^* \to W^*$ the dual bracket of δ_1, that is, if $\delta_1(v) = v_1 \otimes v_2$ then $[\lambda, \mu]^*(v) = \lambda(v_1)\mu(v_2)$. If $\{\lambda_j\}_{j=1}^{\dim W}$ is a basis of W^*, define $S_j : W^* \to W \cong W^{**}$ so that

$$[\lambda, \mu]^* = \sum_j S_j(\mu)(\lambda)\lambda_j$$

Theorem 1.4. *Let* $(\mathfrak{n}, \delta)$ *be a 2-step nilpotent Lie bialgebra, then*

- δ_1 *is a Lie coalgebra structure on* W.
- *Consider* $\{\lambda_j\}_{j=1}^{\dim W}$, $\{\mathfrak{z}_i\}_{i=1}^{\dim \mathfrak{z}}$ *basis of* W^* *and* $\mathfrak{z}$, *resp. and* T_i *and* S_j *as before. If in addition* $\delta(\mathfrak{z}) \subset \Lambda^2 \mathfrak{z}$, *then*

$$T_i S_j T_k + T_k S_j T_i = 0 \ \forall i, j, k$$

Example 1.5. In the Heisenberg Lie algebra there is only one T, corresponding to the symplectic form ω, which is non degenerate, so the condition $\delta(z) = 0$ together with the equation $T S_j T = 0 \ \forall j$ clearly implies $\delta_1 = 0$.

Proof. We need to check co-Jacobi for δ_1. If $z \in \mathfrak{z}$ then $\delta(z) \in (\Lambda^2 \mathfrak{n})^{\mathfrak{n}}$ and one can easily check that we always have

$$(\Lambda^2 \mathfrak{n})^{\mathfrak{n}} \subseteq W \wedge \mathfrak{z} \oplus \Lambda^2 \mathfrak{z}$$

that is, $\delta(z)$ has zero component in $\Lambda^2 W$.

On the other hand, if $v \in W$, $\delta(v) \in \Lambda^2 W \oplus W \wedge \mathfrak{z} \oplus \Lambda^2 \mathfrak{z}$ and we can write

$$\delta(v) = \delta_1(v) \text{ Mod } (W \wedge \mathfrak{z} + \Lambda^2 \mathfrak{z})$$

and the component of "$\delta^2(v)$"$= (\delta \otimes 1 - 1 \otimes \delta)(\delta v)$ in $\Lambda^3 W$ can only arise because of δ_1 applied to components of $\delta_1(v)$, since δ applied to elements in $\mathfrak{z}$ give components with at least one "z", that is, components in $\Lambda^2 W \wedge \mathfrak{z} + W \wedge \Lambda^2 \mathfrak{z} + \Lambda^3 \mathfrak{z}$. We conclude that δ_1 satisfies $(\delta_1 \otimes 1 - 1 \otimes \delta_1)(\delta_1 v) = 0$.

The second part is the most interesting, and this part is a generalization of the arguments for the Heisenberg case. Let $u, v \in W$, recall the cocycle condition is

$$\delta[u,v] = [\delta u, v] + [u, \delta v]$$

Let us write

$$\delta u = \delta_1 u + \sum_i D^i(u) \wedge z_i + \delta_W^{\Lambda^2\mathfrak{z}} u$$

(and similarly for δv) where $\delta_W^{\Lambda^2\mathfrak{z}} u$ is the component in $\Lambda^2\mathfrak{z}$ of δu and $D^i : W \to W$ are linear maps describing the $W \wedge \mathfrak{z}$-component. Then $\delta[u,v] =$

$$= \Big[\delta_1 u + \sum_i D^i(u) \wedge z_i + \delta_W^{\Lambda^2\mathfrak{z}} u, v\Big] + \Big[u, \delta_1 v + \sum_i D^i(v) \wedge z_i + \delta_W^{\Lambda^2\mathfrak{z}} v\Big]$$

$$= \Big[\delta_1 u + \sum_i D^i(u) \wedge z_i, v\Big] + \Big[u, \delta_1 v + \sum_i D^i(v) \wedge z_i\Big]$$

$$= [\delta_1 u, v] + \sum_i [D^i(u), v] \wedge z_i + [u, \delta_1 v] + \sum_i [u, D^i(v)] \wedge z_i$$

$$= \underbrace{[\delta_1 u, v] + [u, \delta_1 v]}_{\in W \wedge \mathfrak{z}} + \sum_i \underbrace{([D^i(u), v] + [u, D^i(v)]) \wedge z_i}_{\in \Lambda^2\mathfrak{z}}$$

Since $\mathfrak{n}$ is 2-step nilpotent we know $[u,v] \in \mathfrak{z}$, but we also assume $\delta(\mathfrak{z}) \subseteq \Lambda^2\mathfrak{z}$, so $\delta[u,v] \in \Lambda^2\mathfrak{z}$, hence

$$\begin{cases} 0 & = & [\delta_1 u, v] + [u, \delta_1 v] \\ \delta[u,v] & = & \sum_i ([D^i(u), v] + [u, D^i(v)]) \wedge z_i \end{cases}$$

In particular, the first equality holds. The second identity is not needed now, but it will be used in the proof of the "general construction" theorem. In Sweedler - type notation, $\delta_1 u = u_1 \wedge u_2$, $\delta_1 v = v_1 \wedge v_2$, so we have

$$0 = [u_1, v] \wedge u_2 + u_1 \wedge [u_2, v] + [u, v_1] \wedge v_2 + v_1 \wedge [u, v_2]$$

$$= -u_2 \wedge [u_1, v] + u_1 \wedge [u_2, v] - v_2 \wedge [u, v_1] + v_1 \wedge [u, v_2] \in W \wedge \mathfrak{z}$$

and by (co)antisymmetry of δ_1 (i.e. "$v_1 \wedge v_2$" $= \sum v_1 \otimes v_2$ is an antisymmetric tensor) each of these expressions are antisymmetric under the change of indices $1 \leftrightarrow 2$. Using this antisymmetry in the first and third terms (but leaving untouched the second and fourth ones), we get

$$0 = 2u_1 \wedge [u_2, v] + 2v_1 \wedge [u, v_2] \in W \wedge \mathfrak{z} \cong W \otimes \mathfrak{z}$$

Hence $\forall\ \phi \in \mathfrak{z}^*$, we have the formula

$$u_1 \phi([v, u_2]) = v_1 \phi([u, v_2]) \tag{1}$$

For $v \in W$ and $\phi \in \mathfrak{z}^*$, denote $v_\phi \in W^*$ the element defined by

$$v_\phi(w) := \phi[v, w]$$

In this notation the above formula reads

$$u_1 v_\phi(u_2) = v_1 u_\phi(v_2)$$

For any w, apply $w_\phi \in W^*$ and get

$$w_\phi(u_1) v_\phi(u_2) = w_\phi(v_1) u_\phi(v_2)$$

Recall again $[\ ,\ ]^* := \delta^* : W^* \times W^* \to W^*$ is the transpose of δ_1, so we re-write the formula as

$$[w_\phi, v_\phi]^*(u) = [w_\phi, u_\phi]^*(v)$$

Now we use alternatively the antisymmetry of $[\ ,\ ]^*$ and the above formula and get

$$[w_\phi, v_\phi]^*(u) = -[v_\phi, w_\phi]^*(u) = -[v_\phi, u_\phi]^*(w) = [u_\phi, v_\phi]^*(w)$$

$$= [u_\phi, w_\phi]^*(v) = -[w_\phi, u_\phi]^*(v)$$

hence

$$\boxed{[w_\phi, u_\phi]^*(v) = 0 \text{ for all } u, v, w \in W,\ \phi \in \mathfrak{z}^*}$$

This is almost the end of the proof. Now we simply write this fact using bases. Recall

$$[\cdot,\cdot]^* = \sum_{k=1}^{\dim W} \lambda_k \otimes S_k, \qquad S_i : W^* \to W, \quad S + S^t = 0;$$

$$[\cdot,\cdot] = \sum_{i=1}^{\dim \mathfrak{z}} z_i \otimes T_i, \qquad T_i : W \to W^*, \quad T + T^t = 0.$$

where $\{z_i\}_{i=1}^{\dim \mathfrak{z}}$ is basis of $\mathfrak{z}$ and $\{\lambda_k\}_{i=k}^{\dim W}$ is basis of W^*.
Besides, $u_\phi = \sum_{i=1}^{\dim \mathfrak{z}} \phi(z_i) T_i(u)$, so

$$0 = [u_\phi, v_\phi]^* = \sum_{i=1}^{\dim \mathfrak{z}} \sum_{j=1}^{\dim \mathfrak{z}} \phi(z_i)\phi(z_j)[T_i(u), T_j(v)]^*$$

$$= \sum_{i=1}^{\dim \mathfrak{z}} \sum_{j=1}^{\dim \mathfrak{z}} \sum_{k=1}^{\dim W} \phi(z_i)\phi(z_j)\, T_j(v)\big(S_k(T_i(u))\big)\, \lambda_k$$

$T_j(a)(b) = -T_j(b)(a) \Longrightarrow T_j(v)(S_k(T_i(u))) = -T_j(S_k(T_i(u)))(v)$, so

$$= -\sum_{i=1}^{\dim \mathfrak{z}} \sum_{j=1}^{\dim \mathfrak{z}} \sum_{k=1}^{\dim W} \phi(z_i)\phi(z_j)\, T_j\big(S_k(T_i(u))\big)(v)\, \lambda_k,$$

Since $\{\lambda_k\}$ is a basis

$$0 = \sum_{i,j} \phi(z_i)\phi(z_j)\, T_j\big(S_k(T_i(u))\big)(v) \ \ \forall k$$

But because it holds for all u and v we have

$$0 = \sum_{i,j} \phi(z_i)\phi(z_j) T_j S_k T_i \in \mathrm{Hom}(W, W^*)$$

In particular, if $\{z^i\}$ denotes the dual basis of $\{z_i\}$, taking $\phi = z^{i_0}$ one gets

$$0 = T_{i_0} S_k T_{i_0} \ \forall k, i_0,$$

which is a particular case. Specializing at $\phi = z^{i_0} + z^{j_0}$ (and using the particular case) one can easily get

$$\boxed{0 = T_{j_0} S_k T_{i_0} + T_{i_0} S_k T_{j_0} \ \forall i_0, j_0, k}$$

□

Remark 1.6. Since the Lie algebra structure will be fixed and the cobracket is the unknown, one can look at

$$\text{"}T_i S_j T_k + T_k S_j T_i = 0 \ \forall i, j, k\text{"}$$

as a linear system of equations on the unknown "S". This motivates the following:

Definition 1.7. Keeping notation for $\mathfrak{n} = W \oplus \mathfrak{z}$ and the T_i's, we call $\mathfrak{n}$ a 2-step Lie algebra of **TST** type if the system of equations

$$T_i S T_k + T_k S T_i = 0 \ \forall i, k$$

with unknown $S : W^* \to W$ has only the trivial solution $S = 0$.

Examples of TST algebras are the Heisenberg algebras. Also one can check that $\mathfrak{f}_n$, the free 2-step nilpotent Lie algebra on n-generators, is of TST type, and this is included in a big family of examples that one can build from graphs.

Corollary 1.8. *If $(\mathfrak{n}, \delta)$ is a Lie bialgebra of type TST and $(\Lambda^2 \mathfrak{n})^{\mathfrak{n}} = \Lambda^2 \mathfrak{z}$ then*

- $\delta \mathfrak{z} \subset \Lambda^2 \mathfrak{z}$
- $\delta W \subset W \wedge \mathfrak{z} \oplus \Lambda^2 \mathfrak{z}$,

Proof. The first part is because $\delta \mathfrak{z} \subseteq (\Lambda^2 \mathfrak{n})^{\mathfrak{n}}$ and the second is because the $\Lambda^2 W$-component of $\delta|_W$ is δ_1, that is determined by S_j's verifying the "TST" system of equations. □

1.1 General Construction

Theorem 1.9. *Assume the following data on $\mathfrak{n} = W \oplus \mathfrak{z}$ is given:*

- $\delta_{\mathfrak{z}} : \mathfrak{z} \to \Lambda^2 \mathfrak{z}$ *a Lie coalgebra structure,*
- *a Lie algebra map $\mathbb{D} \colon \mathfrak{z}^* \to \mathrm{End}(W)$, $f \mapsto -\sum_i f(z_i) D^i$ verifying the following*

$$\sum_i T_i(x)(y) \delta_{\mathfrak{z}} z_i = \sum_{i,j} \Big(T_i(D^j(x))(y) + T_i(x)(D^j(y)) \Big) z_i \wedge z_j$$

where $D^i = -\mathbb{D}(z^i)$.

- $\Phi\colon \Lambda^2\mathfrak{z}^* \to W^*$ *a 2-cocycle with values in* W^*

then the map $\delta : \mathfrak{n} \to \Lambda^2\mathfrak{n}$ *defined by*

$$\begin{cases} \delta(z) &= \delta_{\mathfrak{z}}(z) & \textit{if } z \in \mathfrak{z} \\ \delta(v) &= \sum_i D^i(v) \wedge z_i + \Phi^*(v) & \textit{if } v \in W \end{cases}$$

is a Lie bialgebra structure on $\mathfrak{n}$*.*

Proof. Straightforward checking. □

Now TST condition is a useful tool because one can easyly prove the following

Theorem 1.10. *If* $\mathfrak{n}$ *is of type TST and* $(\Lambda^2\mathfrak{n})^{\mathfrak{n}} = \Lambda^2\mathfrak{z}$ *then* **all** *Lie bialgebra structures on* $\mathfrak{n}$ *are as in the previous Theorem.*

Proof. Last corollary says that necesarily $\delta\mathfrak{z} \subset \Lambda^2\mathfrak{z}$ and $\delta W \subset W \wedge \mathfrak{z} \oplus \Lambda^2\mathfrak{z}$. If one write δ in terms of arbitary maps $\delta_{\mathfrak{z}}$, D^j's and Φ, then it is a straightforward checking that co-Jacobi condition implies first and third item of Theorem 1.9 and the 1-cocycle condition implies the second item. □

2 Graph algebras

Let $G = (V, A)$ be an oriented simple graph without loops. The graph algebra $\mathfrak{n}(G)$ associated to a graph G is defined in the following way: for each $i, j \in V$ and $\alpha \in A$ going from i to j we set

$$[e_i, e_j] := \alpha$$

Since the isomorphism class does not depends on the orientation (just change α by $-\alpha$), sometimes we will assume that G is unoriented, but the set of vertices is ordered, so that an edge joining two vertices can be oriented, considering that it goes from the smaller to the bigger.

Examples 2.1.

1. The graph $x \xrightarrow{Z} y$ gives the 3-dimensional Lie algebra with basis $\{x, y, z\}$ and bracket $[x, y] = z$, that is, the Heisenberg algebra $\mathfrak{h}_3$. On the other hand, for $n > 1$ the Heisenberg Lie algebra $\mathfrak{h}_{2n+1}$ is *not* a graph algebra.

2. $\mathfrak{f}_n$: The free 2-step nilpotent Lie algebra is the graph algebra associated to the complete graph K_n.

3. The nontrivial brackets of the Lie algebra associated to the following graph are

$$\begin{tikzcd} & e_2 \arrow[d, "\alpha"] & \\ e_1 \arrow[ur, "\gamma"] \arrow[r, "\beta"'] & e_3 & e_4 \arrow[l, "\rho"] \end{tikzcd}$$

$$[e_1, e_2] = \gamma, \quad [e_1, e_3] = \beta,$$
$$[e_2, e_3] = \alpha, \quad [e_4, e_3] = \rho.$$

Notice that the center of a graph algebra has natural basis consisting on the arrows and the isolated vertices. If we assume that the graph does not have isolated vertices then we have canonical decomposition and basis:

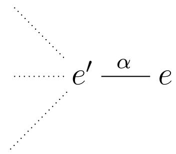
$$W = \bigoplus_{e\in V} ke,\ \mathfrak{z} = \bigoplus_{\alpha\in A} k\alpha$$

and $\mathfrak{n}(G) = W \oplus \mathfrak{z}$ is a 2-step nilpotent with center $\mathfrak{z}$ and linear complement W.

We begin by characterizing the condition $(\Lambda^2\mathfrak{n})^{\mathfrak{n}} = \Lambda^2\mathfrak{z}$. Recall that for a vertex e, the degree - or valency- $|e|$ is the number of edges incident to e.

Lemma 2.2. *Consider a graph $G = (V, A)$ such that there exists $e \in V$ with $|e| = 1$ then there exists an element $0 \neq \omega \in (W \wedge \mathfrak{z})^{\mathfrak{n}}$.*

Proof. Let α be the unique edge joining e with e',

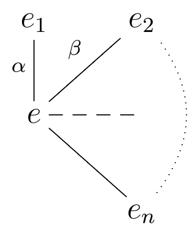


If we consider $\omega := e \wedge \alpha \in W \wedge \mathfrak{z}$ then $\mathrm{ad}_{e'}\omega = \pm\alpha \wedge \alpha = 0$, and clearly also $\mathrm{ad}_{e''}\omega = 0$ for any other $e'' \in V$. □

A more involved proof is needed for the following Lemma:

Lemma 2.3. *Let $\omega \in (W \wedge \mathfrak{z})^{\mathfrak{n}}$, if $\omega = \sum_{e\in V,\alpha\in A} \lambda_{e,\alpha} e \wedge \alpha$, then $\lambda_{e,\alpha} = 0$ for each $e \in V$ such that $|e| \geq 2$.*

Proof. Consider $e \in V$ a vertex with $|e| \geq 2$.



Denote $e' = e_1$ and $e'' = e_2$ in the drawing, i.e. e' and e'' are two different vertices incident to e with corresponding edges α and β. A general element in $W \wedge \mathfrak{z}$ will be of the form

$$\begin{aligned}
\omega \quad = \quad & ae \wedge \alpha && +be \wedge \beta && +e \wedge \Big(\sum_{\alpha_i\neq\alpha,\beta} \lambda_i\alpha_i\Big) \\
& +a'e' \wedge \alpha && +b'e' \wedge \beta && +e' \wedge \Big(\sum_{\alpha_i\neq\alpha,\beta} \mu_i\alpha_i\Big) \\
& +a''e'' \wedge \alpha && +b''e'' \wedge \beta && +e'' \wedge \Big(\sum_{\alpha_i\neq\alpha,\beta} \nu_i\alpha_i\Big) \\
& + \sum_{e'''\neq e,e',e''} a_{e'''}e''' \wedge \alpha && + \sum_{e'''\neq e,e',e''} b_{e'''}e''' \wedge \beta && + \sum_{\substack{e'''\neq e,e',e'' \\ \alpha_i\neq\alpha,\beta}} \lambda_{i,e'''}e''' \wedge \alpha_i
\end{aligned}$$

for some $a, a', a''a_i, \mu_i, \nu_i, \lambda_{i,e'''} \in k$. Since ω is supposed to be invariant, we have

$$
\begin{array}{rcccc}
0 = \mathrm{ad}_e\omega & = & 0 & +0 & +0 \\
& & +a'\alpha\wedge\alpha & +b'\alpha\wedge\beta & +\alpha\wedge(\sum\limits_{\alpha_i\neq\alpha,\beta}\mu_i\alpha_i) \\
& & +a''\beta\wedge\alpha & +b''\beta\wedge\beta & +\beta\wedge(\sum\limits_{\alpha_i\neq\alpha,\beta}\nu_i\alpha_i) \\
& & +\sum\limits_{e'''\neq e,e',e''} a_{e'''}[e,e''']\wedge\alpha & +\sum\limits_{e'''\neq e,e',e''} b_{e'''}[e,e''']\wedge\beta & +\sum\limits_{\substack{e'''\neq e,e',e''\\ \alpha_i\neq\alpha,\beta}}\lambda_{i,e'''}[e,e''']\wedge\alpha_i
\end{array}
$$

then, in particular $b' = a''$ $\quad(*)$.
Analogously, compute $0 = \mathrm{ad}_{e'}\omega =$

$$
\begin{array}{rccc}
= & -a\alpha\wedge\alpha & -b\alpha\wedge\beta & -\alpha\wedge(\sum\limits_{\alpha_i\neq\alpha,\beta}\lambda_i\alpha_i) \\
& +0 & +0 & +0 \\
& +a''[e',e'']\wedge\alpha & +b''[e',e'']\wedge\beta & +[e',e'']\wedge(\sum\limits_{\alpha_i\neq\alpha,\beta}\nu_i\alpha_i) \\
& +\sum\limits_{e'''\neq e,e',e''} a_{e'''}[e',e''']\wedge\alpha & +\sum\limits_{e'''\neq e,e',e''} b_{e'''}[e',e''']\wedge\beta & +\sum\limits_{\substack{e'''\neq e,e',e''\\ \alpha_i\neq\alpha,\beta}}\lambda_{i,e'''}[e',e''']\wedge\alpha_i \\
= & & -b\alpha\wedge\beta & -\alpha\wedge(\sum\limits_{\alpha_i\neq\alpha,\beta}\lambda_i\alpha_i) \\
& +a''[e',e'']\wedge\alpha & +b''[e',e'']\wedge\beta & +[e',e'']\wedge(\sum\limits_{\alpha_i\neq\alpha,\beta}\nu_i\alpha_i) \\
& +\sum\limits_{e'''\neq e,e',e''} a_{e'''}[e',e''']\wedge\alpha & +\sum\limits_{e'''\neq e,e',e''} b_{e'''}[e',e''']\wedge\beta & +\sum\limits_{\substack{e'''\neq e,e',e''\\ \alpha_i\neq\alpha,\beta}}\lambda_{i,e'''}[e',e''']\wedge\alpha_i
\end{array}
\tag{1}
$$

Hence, $b = 0$.
In the same way $0 = \mathrm{ad}_{e''}\omega =$

$$
\begin{array}{rccc}
= & -a\beta\wedge\alpha & & -\beta\wedge(\sum\limits_{\alpha_i\neq\alpha,\beta}\lambda_i\alpha_i) \\
& +a'[e'',e']\wedge\alpha & +b'[e'',e']\wedge\beta & +[e'',e']\wedge(\sum\limits_{\alpha_i\neq\alpha,\beta}\nu_i\alpha_i) \\
& +\sum\limits_{e'''\neq e,e',e''} a_{e'''}[e'',e''']\wedge\alpha & +\sum\limits_{e'''\neq e,e',e''} b_{e'''}[e'',e''']\wedge\beta & +\sum\limits_{\substack{e'''\neq e,e',e''\\ \alpha_i\neq\alpha,\beta}}\lambda_{i,e'''}[e'',e''']\wedge\alpha_i
\end{array}
\tag{2}
$$

Hence, $a = 0$. To continue, we will consider cases $i)$ and $ii)$ as follows:

$i)$ Suppose e' and e'' are not joined by any edge in the graph, so $[e', e''] = 0$, then (1) equals

$$
\begin{array}{rcl}
0 & = & \Big(\sum\limits_{\alpha_i\neq\alpha,\beta}\lambda_i\alpha_i + \sum\limits_{e'''\neq e,e',e''} a_{e'''}[e',e''']\Big)\wedge\alpha \\
& & +\sum\limits_{e'''\neq e,e',e''} b_{e'''}[e',e''']\wedge\beta \qquad +\sum\limits_{\substack{e'''\neq e,e',e''\\ \alpha_i\neq\alpha,\beta}}\lambda_{i,e'''}[e',e''']\wedge\alpha_i
\end{array}
\tag{1'}
$$

Notice in the previous equation that all the terms belong to different components, so $(1')$ implies $\lambda_i = 0$ for all i except those corresponding to α_i incident to e'.

In the same way, (2) implies $\lambda_i = 0$ for all i except those corresponding to α_i incident to e''. But there is no edge between e' and e'', hence all $\lambda_i = 0$.

$ii)$ Suppose e' and e'' are joined by an edge γ, so $[e', e''] = \gamma$. Equation 1 reads in this case

$$\begin{aligned} 0 \;=\;& \left(\sum_{\alpha_i \neq \alpha,\beta} \lambda_i\alpha_i + a''\gamma + \sum_{e'''\neq e,e',e''} a_{e'''}[e', e''']\right) \wedge \alpha \\ & +b''\gamma\wedge\beta \qquad\qquad +\gamma \wedge (\sum_{\alpha_i\neq\alpha,\beta} \nu_i\alpha_i) \\ & + \sum_{e'''\neq e,e',e''} b_{e'''}[e', e'''] \wedge \beta \qquad + \sum_{\substack{e'''\neq e,e',e'' \\ \alpha_i\neq\alpha,\beta}} \lambda_{i,e'''}[e', e'''] \wedge \alpha_i \end{aligned} \tag{1''}$$

Looking at the terms with common factor α, we see that $\lambda_i = 0$ for all i except those corresponding to α_i incident to e'. A similar computation interchanging e' with e'' says $\lambda_i = 0$ for all i except those corresponding to α_i incident to e''.

Resuming, if we call $c = b' = a''$ and $\lambda = \lambda_\gamma$, ω is of the following form

$$\begin{aligned} \omega \;=\;& \qquad\qquad\qquad\qquad e\wedge\lambda\gamma \\ & +a'e'\wedge\alpha \qquad +ce'\wedge\beta \qquad +e'\wedge(\sum_{\alpha_i\neq\alpha,\beta}\mu_i\alpha_i) \\ & +ce''\wedge\alpha \qquad +b''e''\wedge\beta \qquad +e''\wedge(\sum_{\alpha_i\neq\alpha,\beta}\nu_i\alpha_i) \\ & +\sum_{e'''\neq e,e',e''} a_{e'''}e'''\wedge\alpha + \sum_{e'''\neq e,e',e''} b_{e'''}e'''\wedge\beta + \sum_{\substack{e'''\neq e,e',e'' \\ \alpha_i\neq\alpha,\beta}} \lambda_{i,e'''}e'''\wedge\alpha_i \end{aligned}$$

We go back to equations (1) and (2), we have

$$\begin{aligned} 0 \;=\;& \qquad\qquad\qquad -\lambda\alpha\wedge\gamma \\ & +c\gamma\wedge\alpha \qquad +b''\gamma\wedge\beta \qquad +\gamma\wedge(\sum_{\alpha_i\neq\alpha,\beta}\nu_i\alpha_i) \\ & +\sum_{e'''\neq e,e',e''} a_{e'''}[e',e''']\wedge\alpha + \sum_{e'''\neq e,e',e''} b_{e'''}[e',e''']\wedge\beta + \sum_{\substack{e'''\neq e,e',e'' \\ \alpha_i\neq\alpha,\beta}} \lambda_{i,e'''}[e',e''']\wedge\alpha_i \end{aligned} \tag{1}$$

and

$$\begin{aligned} 0 \;=\;& \qquad\qquad\qquad\qquad -\lambda\beta\wedge\gamma \\ & -a'\gamma\wedge\alpha \qquad -c\gamma\wedge\beta \qquad -\gamma\wedge(\sum_{\alpha_i\neq\alpha,\beta}\nu_i\alpha_i) \\ & +\sum_{e'''\neq e,e',e''} a_{e'''}[e'',e''']\wedge\alpha + \sum_{e'''\neq e,e',e''} b_{e'''}[e'',e''']\wedge\beta + \sum_{\substack{e'''\neq e,e',e'' \\ \alpha_i\neq\alpha,\beta}} \lambda_{i,e'''}[e'',e''']\wedge\alpha_i \end{aligned} \tag{2}$$

We look at the terms with $\alpha\wedge\gamma$ in (1) and $\beta\wedge\gamma$ in (2) and obtain

$$\lambda + c = 0 \text{ and } \lambda - c = 0$$

Hence, $\lambda = 0$, so $\lambda_{e,\alpha_i} = 0$ for all $\alpha_i \in A$. □

As a corollary we can prove the following characterization:

Theorem 2.4. *For a graph algebra (without isolated vertices),* $(\Lambda^2\mathfrak{n})^\mathfrak{n} = \Lambda^2\mathfrak{z}$ *if and only if* $|e| \geq 2$ *for all* $e \in V$.

Proof. Lemmas above show that $(\mathfrak{n} \wedge \mathfrak{z})^\mathfrak{n} = \Lambda^2\mathfrak{z}$, we will show that this is enough to conclude $(\Lambda^2\mathfrak{n})^n = \Lambda^2\mathfrak{z}$.

From the vector space decomposition $\mathfrak{n} = W \oplus \mathfrak{n}$ we get, as usual, the vector space decomposition

$$\Lambda^2\mathfrak{n} = \Lambda^2 W \ \oplus \ W \wedge \mathfrak{z} \ \oplus \ \Lambda^2\mathfrak{z}$$

This *is not* an $\mathfrak{n}$-module decomposition, but if $v \in \mathfrak{n}$, then it is clear that

$$\mathrm{ad}_v(\Lambda^2 W) \subseteq W \wedge \mathfrak{z}, \ \mathrm{ad}_v(W \wedge \mathfrak{z}) \subseteq \Lambda^2\mathfrak{z} \text{ and } \mathrm{ad}_v(\Lambda^2\mathfrak{z}) = 0$$

So, if $u = u_2 + u_1 + u_0 \in \Lambda^2\mathfrak{n}$, with $u_2 \in \Lambda^2 W$, $u_1 \in W \wedge \mathfrak{z}$ and $u_0 \in \Lambda^2\mathfrak{z}$, then

$$u_2 + u_1 + u_0 \in (\Lambda^2\mathfrak{n})^\mathfrak{n} \iff u_i \in (\Lambda^2\mathfrak{n})^\mathfrak{n} : i = 1, 2$$

Hence, it is enough to see that $\Lambda^2 W \cap (\Lambda^2\mathfrak{n})^\mathfrak{n} = 0$. This fact is true for any 2-step nilpotent Lie algebra and the argument is a computation on basis, we include it here for completeness.

Let $\{v_i\}_{i\in I}$ be an ordered basis of W, $\{z_k\}_{k\in K}$ a basis of $\mathfrak{z}$, let

$$[v_i, v_j] = \sum_{k\in K} T_k(v_i)(v_j) z_k = \sum_k c_{ij}^k z_k$$

be the structure constants, and consider the basis $\{v_i \wedge v_j : i, j \in I, i < j\}$ of $\Lambda^2 W$. Let v_{i_0} be a basis element and write a general element of $\Lambda^2 W$ in the form

$$u = \sum_{i,j} u_{ij} v_i \wedge v_j \in \Lambda^2 W, \ \text{ with } u_{ij} = -u_{ji}$$

If we also assume that $u \in (\Lambda^2\mathfrak{n})^\mathfrak{n}$, then

$$0 = \mathrm{ad}_{v_{i_0}}(u) = \sum_{i,j} \left(u_{ij}[v_{i_0}, v_i] \wedge v_j + u_{ij} v_i \wedge [v_{i_0}, v_j]\right)$$

$$= \sum_{i,j,k} u_{ij}\left(c_{i0,i}^k z_k \wedge v_j + c_{i_0,j}^k v_i \wedge z_k\right)$$

$$= \sum_k \left(\sum_{i,j} u_{ij}(-c_{i0,i}^k v_j + c_{i_0,j}^k v_i)\right) \wedge z_k \qquad \in W \wedge \mathfrak{z} \cong W \otimes \mathfrak{z}$$

since z_k is a basis of $\mathfrak{z}$ we have

$$\sum_{i,j} u_{ij}(-c_{i0,i}^k v_j + c_{i_0,j}^k v_i) = 0 \ \ \forall k$$

But

$$0 = \sum_{i,j} u_{ij}(-c_{i0,i}^k v_j + c_{i_0,j}^k v_i) = \sum_{i,j} -u_{ij} c_{i0,i}^k v_j + \sum_{i,j} u_{ij} c_{i_0,j}^k v_i$$

$$= \sum_{i,j} -u_{ij} c^k_{i0,i} v_j + \sum_{j,i} u_{ji} c^k_{i_0,i} v_j$$
$$= \sum_j \left(\sum_i (-u_{ij} + u_{ji}) c^k_{i_0,i} \right) v_j$$
$$= 2 \sum_j \left(\sum_i u_{ji} c^k_{i_0,i} \right) v_j$$

then $\boxed{\sum_i u_{ji} c^k_{i_0,i} = 0 \ \ \forall j, \forall k}$. From this equality we see that the element

$$w_j := \sum_i u_{ji} v_i$$

satisfies $[v_{i_0}, w_j] = 0$ for all v_{i_0}; that is, w_j belongs to the center, so $w_j \in W \cap \mathfrak{z} = 0$. Hence, $u_{ij} = 0$ for all i, j, so $u = \sum_{i,j} u_{ij} v_i \wedge v_j = 0$ and the proof is complete. $\square$

Remark 2.5. The argument we have used in the previous proof is valid for *any* 2-step nilpotent Lie algebra. In particular, $(\Lambda^2 \mathfrak{n})^{\mathfrak{n}} = (\mathfrak{z} \wedge \mathfrak{n})^{\mathfrak{n}}$ for any 2-step nilpotent Lie algebra.

2.1 Graph algebras and the TST equations

For a graph $G = (V, A)$ with vertices V and arrows A, without isolated vertices, we use the canonical basis of the center $\{z_i\}_{i=1}^{\dim \mathfrak{z}} := \{\alpha\}_{\alpha \in A}$. The system of equations with unknown antisymmetric map $S : W^* \to W$ is of the form

$$T_\alpha S T_\beta + T_\beta S T_\alpha = 0 \ \forall \alpha, \beta \in A$$

Fix $V = \{e_1, \ldots, e_n\}$ the set of vertices, it is a basis of W by definition; let $\{e_1^*, \ldots, e_n^*\}$ be the dual basis. It is easy to see that for each $\alpha \in A$ joining e_i with e_j, with $i < j$ we have

$$T_\alpha(e_i) = e_j^*, \ T_\alpha(e_j) = -e_i^*, \ T_\alpha(e_k) = 0 \ \forall k \neq i, j$$

In matrix notation $[T_\alpha] = E_{j,i} - E_{i,j}$. The following is a translation of the TST condition in graph language:

Theorem 2.6. *Let $\mathfrak{n}$ be a 2-steps nilpotent Lie algebra arising from a graph $G = (V, A)$, $i, j \in V$ and S an antisymmetric solution of the system $T_\alpha S T_\beta + T_\beta S T_\alpha = 0 \ \forall \alpha, \beta \in A$.*

1. *If there exists an edge α joining i and j, then $S_{i,j} = 0$.*

$$i \xrightarrow{\alpha} j$$

2. *If there are two edges α and β and four vertices i, i', j, j' such that α joins i and i' and β joins j and j' with $\{i, i'\} \cap \{j, j'\} = \emptyset$ then $S_{i,j} = 0$.*

$$\begin{tikzcd} i \arrow[r, dotted, no head] \arrow[d, "\alpha"] & j \arrow[d, "\beta"] \\ i' \arrow[r, dotted, no head] & j' \end{tikzcd}$$

Proof. Consider the equation $T_\alpha S T_\alpha = 0$ for a given $\alpha \in A$, where α joins e_i with e_j.

$$(T_\alpha S T_\alpha)_{i,j} = \sum_{k,\ell} (T_\alpha)_{i,k} (S)_{k,\ell} (T_\alpha)_{\ell,j} = (T_\alpha)_{i,j} (S)_{j,i} (T_\alpha)_{i,j} = (S)_{j,i}$$

This shows 1. Next suppose we have two edges α and β such that α joins i and i', and β joins j and j' with $\{i, i'\} \cap \{j, j'\} = \emptyset$. Writing the TST- equation for these α, β,

$$(T_\alpha S T_\beta + T_\beta S T_\alpha)_{i',j'} = \sum_{k,\ell} (T_\alpha)_{i',k} S_{k,\ell} (T_\beta)_{\ell,j'} + \sum_{k,\ell} (T_\beta)_{i',k} S_{k,\ell} (T_\alpha)_{\ell,j'}$$

$$= (T_\alpha)_{i',i} S_{i,j} (T_\beta)_{j,j'} = \pm S_{i,j},$$

so $S_{i,j} = 0$. $\square$

Examples

Remark 2.7. If we are interested in algebras with $(\Lambda^2 \mathfrak{n})^{\mathfrak{n}} = \Lambda^2 \mathfrak{z}$ then we must look at graphs with $|e| \geq 2$ for all $e \in V$. But if this is the case, then any pair i, j in V satisfies either condition 1 or 2 of the previous proposition, so we have the following corollary:

Corollary 2.8. *If $|e| \geq 2$ for all $e \in V$ then $(\Lambda^2 \mathfrak{n})^{\mathfrak{n}} = \Lambda^2 \mathfrak{z}$ and also $\mathfrak{n}$ is of TST type, as a consequence, any cobracket structure on $\mathfrak{n}$ satisfies*

$$\delta \mathfrak{z} \subseteq \Lambda^2 \mathfrak{z}, \quad \delta W \subseteq W \wedge \mathfrak{z} \oplus \Lambda^2 \mathfrak{z},$$

and it is of the form as in Theorem 1.9.

Remark 2.9. In the case where a graph has (all vertices with) valency bigger or equal to 2, even if the above corollary shows a big simplification of the structure constants, in order to find all possible data as in Theorem 1.9 we still have to solve nonlinear equations, for instance, the data includes a Lie algebra structure on $\mathfrak{z}^*$. We present next a family of Lie bialgebras that include the ones where the 2-cocycle δ is a coboundary where this part of the data is trivial.

3 Nearly Coboundary Lie bialgebras

Recall that a Lie bialgebra $(\mathfrak{n}, \delta)$ is called **coboundary** if there exists $r \in \Lambda^2\mathfrak{n}$ such that

$$\delta(x) = \mathrm{ad}_x(r)\ \forall x \in \mathfrak{n}$$

In any coboudary bialgebra we have $\delta(\mathfrak{z}) = 0$. This motivates the following definition

Definition 3.1. A Lie bialgebra $(\mathfrak{n}, \delta)$ will be called **nearly coboundary** if $\delta|_{\mathfrak{z}} \equiv 0$.

Example 3.2. Let $\delta : \mathfrak{n} \to \Lambda^2\mathfrak{n}$ a linear map satisfying $\delta(W) \subset \Lambda^2\mathfrak{z}$ and $\delta(\mathfrak{z}) = 0$, that is, the only nontrivial component is of the form

$$\delta(e_i) = \sum_{\alpha,\beta\in A} \mu_i^{\alpha,\beta} \alpha \wedge \beta \quad (e_i \in V)$$

with arbitrary coefficients $\mu_i^{\alpha,\beta} \in K$ verifing only $\mu_i^{\alpha,\beta} = -\mu_i^{\beta,\alpha}$ for all i, α, β. Then δ endows $\mathfrak{n}$ of a nearly coboundary Lie bialgebra structure. Notice that there are $|V|\frac{|A|(|A|-1)}{2}$ free parameters, while for a coboundary Lie bialgebra structure with $\delta(W) \subset \Lambda^2\mathfrak{z}$ we need an element

$$r = \sum_{i\in V,\alpha\in A} r_{ij} e_i \wedge \alpha \in W \wedge \mathfrak{z}$$

In order to give r we need $|V||A|$ parameters; if $|A| > 3$ then $|V|\frac{|A|(|A|-1)}{2} > |V||A|$, so in particular there is a lot of non coboundary Lie algebras in this family of examples.

As particular cases, for $G = C_n$ we have $|V| = n$ and $|A| = n$; for $G = K_n$, $|V| = n$ and $|A| = n(n-1)/2$. Except $n = 3$, there are a lot of nearly coboundary Lie bialgebras of this type that are not coboundary. In the table we write the numbers $|V|\frac{|A|(|A|-1)}{2}$ and $|V||A|$ in general and for small n:

	C_n		K_n	
	$\lvert V\rvert\lvert A\rvert$	$\lvert V\rvert\frac{\lvert A\rvert(\lvert A\rvert-1)}{2}$	$\lvert V\rvert\lvert A\rvert$	$\lvert V\rvert\frac{\lvert A\rvert(\lvert A\rvert-1)}{2}$
n	n^2	$\frac{n^2(n-1)}{2}$	$\frac{n^2(n-1)}{2}$	$\frac{1}{8}n^2(n+1)(n-1)(n-2)$
3	9	9	9	9
4	16	24	24	60
5	25	50	50	225
6	36	90	90	630

Now we will study conditions of Theorem 1.9 for nearly coboundary bialgebra structures on graph algebras $\mathfrak{n} = \mathfrak{n}(G)$.

Lemma 3.3. *Let G be a graph with $|e| \geq 2\ \forall e \in V$, $\mathfrak{n} = \mathfrak{n}(G)$, $\delta : \mathfrak{n} \to \Lambda^2\mathfrak{n}$ a Lie bialgebra structure. Assume that δ is nearly coboundary, so that $\delta(z) = 0$ if $z \in \mathfrak{z}$. If we write (as in Theorem 1.9)*

$$\delta(v) = \sum_{\alpha\in A} D_\alpha(v) \wedge \alpha + \Phi^*(v) \qquad (v \in W)$$

then $D_\alpha D_\beta = D_\beta D_\alpha$ for all edges α, β.

Proof. We know that all bialgebra structures are as in Theorem 1.9, so they must satisfy

$$\sum_{\alpha,\in A} D_\alpha(e_i) \wedge \delta(\alpha) = \sum_{\alpha,\beta\in A} D_\alpha(D_\beta(e_i)) \wedge \alpha \wedge \beta$$

But because $\delta(\mathfrak{z}) = 0$ we have

$$0 = \sum_{\alpha,\beta\in A} D_\alpha(D_\beta(e_i)) \wedge \alpha \wedge \beta = \sum_{\alpha<\beta\in A} (D_\alpha D_\beta - D_\beta D_\alpha)(e_i) \wedge \alpha \wedge \beta$$

So $D_\alpha D_\beta = D_\beta D_\alpha$. □

From the above lemma, we see that a typical situation will be when the D_α's are simultaneously diagonalizable. In next subsection we study this particular case.

3.1 Nearly coboundary bialgebras with diagonal D_α's

In this subsection we suppose that all the $\{D_\alpha : \alpha \in A\}$ are simultaneously diagonalizable, and moreover, that the set of vertices $\{e_i : 1 \leq i \leq n\} = V$ is a basis of eigenvectors. We denote by $\lambda_{i,\alpha}$ the corresponding eigenvalues, i.e., $\forall i, \alpha$

$$D_\alpha(e_i) = \lambda_{i,\alpha} e_i$$

Remark 3.4. The assumption that the set of vertices are eigenvalues is not a lost of generality in the family of free 2-step nilpotent algebras.

Proposition 3.5. *Let α_0 be an edge joining i_0 with j_0, then*

$$\lambda_{i_0,\alpha} = -\lambda_{j_0,\alpha} \; \forall \alpha \neq \alpha_0$$

Proof. From the cocycle condition, the assumption $\delta(\mathfrak{z}) = 0$ and $\delta_W(e_{i_0}) = \sum_{\alpha\in A} D_\alpha(e_{i_0}) \wedge \alpha$, we have

$$\begin{aligned} 0 = \delta(\alpha_0) = \delta([e_{i_0}, e_{j_0}]) &= [\delta(e_{i_0}], e_{j_0}] + [e_{i_0}, \delta(e_{j_0})] \\ &= \Big[\sum_{\alpha\in A} D_\alpha(e_{i_0}) \wedge \alpha, e_{j_0}\Big] + \Big[e_{i_0}, \sum_{\alpha\in A} D_\alpha(e_{j_0}) \wedge \alpha\Big] \\ &= \Big[\sum_{\alpha\in A} \lambda_{i_0,\alpha} e_{i_0} \wedge \alpha, e_{j_0}\Big] + \Big[e_{i_0}, \sum_{\alpha\in A} \lambda_{j_0,\alpha} e_{j_0} \wedge \alpha\Big] \\ &= \alpha_0 \wedge \Big(\sum_{\alpha\in A} \lambda_{i_0,\alpha} + \lambda_{j_0,\alpha}\Big)\alpha \end{aligned}$$

then each coefficient $\lambda_{i_0,\alpha} + \lambda_{j_0,\alpha} = 0$ for all $\alpha \neq \alpha_0$. □

Example 3.6. For the graph

$$\begin{tikzcd} & e_2 \arrow[dr, "\beta"] & \\ e_1 \arrow[ur, "\alpha"] & & e_3 \arrow[ll, "\gamma"] \end{tikzcd}$$

we get $\mathfrak{f}_3$, the free 2-step nilpotent Lie algebra on 3 generators $\{e_1, e_2, e_3\}$. The complete list of Lie bialgebra structures on $\mathfrak{f}_3$ such that $\delta(\mathfrak{z}) = 0$ and diagonal D_{α_i} for all $i = 1, 2, 3$ is

$$\delta(e_1) = e_1 \wedge (a\alpha + b\beta + c\gamma) + \omega_1 =: e_1 \wedge A_1 + \omega_1$$

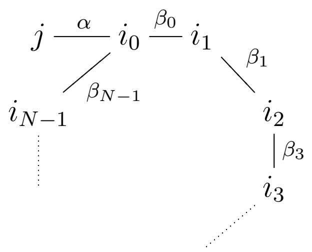
$$\delta(e_2) = e_2 \wedge (a\alpha - b\beta - c\gamma) + \omega_2 =: e_2 \wedge A_2 + \omega_2$$
$$\delta(e_3) = e_3 \wedge (-a\alpha - b\beta + c\gamma) + \omega_3 =: e_3 \wedge A_3 + \omega_3$$

for any $a, b, c \in k$, $\omega_i \in \Lambda^2\mathfrak{z}$ satisfying co - Jacobi condition:

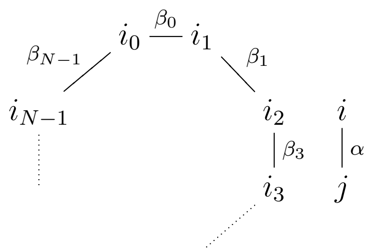
$$\omega_i \wedge A_i = 0, \quad i = 1, 2, 3$$

For highly connected graphs the situation is even more favorable; repeatedly using Proposition 3.5 one can prove the following:

Proposition 3.7. *Consider a graph $G = (V, A)$, a vertex $i_0 \in V$ and $\alpha \in A$. If there exists a (non necesarily oriented) loop in G of the form*



or



then $\lambda_{i_0,\alpha} = (-1)^N \lambda_{i_0,\alpha}$. If N is odd then $\lambda_{i_0,\alpha} = 0$

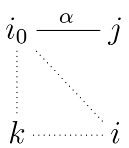
As a corollary we have

Corollary 3.8. *If $n \geq 4$ and δ is a bialgebra structure on $\mathfrak{f}_n$ satisfying $\delta(\mathfrak{z}) = 0$ and D_α diagonalizable for all α, then*

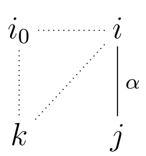
$$\delta(W) \subset \Lambda^2\mathfrak{z},$$

that is, all D_α are necessarily zero and $\delta(e_i)$ is of the form $\delta(e_i) = \omega_i$ with arbitrary $\omega_i \in \Lambda^2\mathfrak{z}$. In other words, δ is as in Example 3.2.

Proof. if i_0 is a vertex and α an edge, in the complete graph on n vertices with $n \geq 4$ we are always in one of the following situations:



or



, so $\lambda_{i_0,\alpha} = 0 \ \forall i_0, \alpha$.

□

We finish exhibiting examples of non diagonalizable D_i's in $\mathfrak{f}_3$. We do not know if there are similar nontrivial examples in $\mathfrak{f}_n$ for $n \geq 4$.

3.2 Non-diagonalizable D's in $\mathfrak{f}_3$

Fix the notation for vertices and arrows:

$$\begin{tikzcd} & e_2 \arrow[dr, "\beta"] & \\ e_1 \arrow[ur, "\alpha"] & & e_3 \arrow[ll, "\gamma"] \end{tikzcd}$$

Since $[D_{\alpha_i}, D_{\alpha_j}] = 0$ for all $\alpha_i, \alpha_j \in \{\alpha, \beta, \gamma\}$, we may always assume that there exists a basis where all D_{α_i} are simultaneously upper triangular, that is, of the form $\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$.

But also, if one of them has three different elements in the diagonal, then all of them are diagonalizable, and we are in the previous case. So we may assume that all of them have some multiplicity on the eigenvalues. We split in two cases: multiplicity 2 and multiplicity 3. In multiplicity 3 there are two possibilities: there is a D with a single Jordan block of size 3, or the maximal size of the Jordan block is 2. Writing the first D in Jordan form and using that the other D's commute with this one we arrive at the following cases:

Multiplicity 2:

$$D_\alpha = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda' \end{pmatrix},\ D_\beta = \begin{pmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & c \end{pmatrix},\ D_\gamma = \begin{pmatrix} \mu & \nu & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \tau \end{pmatrix} \qquad (I)$$

where $\lambda \neq \lambda'$, and multiplicity 3:

$$D_\alpha = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix},\ D_\beta = \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix},\ D_\gamma = \begin{pmatrix} \mu & \nu & \rho \\ 0 & \mu & \nu \\ 0 & 0 & \mu \end{pmatrix} \qquad (II)$$

or

$$D_\alpha = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix},\ D_\beta = \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix},\ D_\gamma = \begin{pmatrix} \mu & \nu & \rho \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix} \qquad (III)$$

From the matrix parameters introduce the elements $A, B, C, D \in W$

$$A := \lambda\alpha + a\beta + \mu\gamma,\ B := \alpha + b\beta + \nu\gamma,\ C := c\beta + \rho\gamma,\ D := \lambda'\alpha + c\beta + \tau\gamma$$

Take elements $\omega_i \in \Lambda^2\mathfrak{z}$ $(i = 1, 2, 3)$ and define

$$\delta(e_i) := D_\alpha(e_i) \wedge \alpha + D_\beta(e_i) \wedge \beta + D_\gamma(e_i) \wedge \gamma + \omega_i$$

In all cases we have

$$\begin{aligned} \delta(e_1) &= e_1 \wedge A + \omega_1 \\ \delta(e_2) &= e_2 \wedge A + e_1 \wedge B + \omega_2 \end{aligned}$$

and $\delta(e_3)$, depending on cases, is equal to

$$\delta(e_3) = \begin{cases} e_3 \wedge D + \omega_3 & (I) \\ e_3 \wedge A + e_2 \wedge B + e_1 \wedge C + \omega_3 & (II) \\ e_3 \wedge A + e_1 \wedge C + \omega_3 & (III) \end{cases}$$

The restriction given by the co-Jacobi identity are

$$\omega_1 \wedge A = 0 = \omega_2 \wedge A + \omega_1 \wedge B$$

and, depending on cases

$$\begin{aligned} (I) \quad 0 &= \omega_3 \wedge D, \\ (II) \quad 0 &= \omega_3 \wedge A + \omega_2 \wedge B + \omega_1 \wedge C, \\ (III) \quad 0 &= \omega_3 \wedge A + \omega_1 \wedge C. \end{aligned}$$

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