Coulomb Blockade in Hierarchical Quantum Hall Droplets

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Abstract. The degeneracy of energy levels in a quantum dot of Hall fluid, leading to conductance peaks, can be readily derived from the partition functions of conformal field theory. Their complete expressions can be found for Hall states with both Abelian and non-Abelian statistics, upon adapting known results for the annulus geometry. We analyze the Abelian states with hierarchical filling fractions, $\nu = m/(mp \pm 1)$, and find a non trivial pattern of conductance peaks. In particular, each one of them occurs with a characteristic multiplicity, that is due to the extended symmetry of the m-folded edge. Experimental tests of the multiplicity can shed more light on the dynamics of this composite edge.

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1. Introduction

Among the recently proposed experimental tests of the quantum Hall effect [1], the study of Coulomb blockade conductance peaks has been proposed in [2][3][4]. One considers an isolated droplet of Hall fluid that is formed in a bar-shaped sample between two constrictions, in the limit of strong quasiparticle backscattering. In this regime, electrons can tunnel into the droplet when energy states are led to be degenerate, either by changing the area of the dot by means of a side modulation gate or by tuning the magnetic field.

Given that the low energy levels of Hall droplets, the edge excitations [5], are described by conformal field theory (CFT) [6], the level deformation and degeneracy can be obtained from the analysis of their known Hilbert spaces. The proper sector in these spaces is identified by the selection rules (the so-called fusion rules [6]) for the addition of electrons to the ground state. The presence of static quasi-particles in the interior of the droplet selects different sectors as dictated by the corresponding fusion rules.

The following results were found in Refs.[3][4]:

i) In the simplest Laughlin Hall states [7], described by the chiral Luttinger liquid (chiral compactified free boson CFT) [5], the peaks in the conductance as a function of the area S are equally spaced by $\Delta S = e/n_o$, where n_o is the electron density. The spacing is the same for any number of quasi-particles in the bulk of the droplet.

ii) In the Read-Rezayi states [8], with $\nu = 2 + k/(k+2)$, described by the chiral boson coupled to non-trivial neutral excitations of the $\mathbb{Z}_k$-Parafermion CFT [6], the spacings between peaks are modulated: a group of k equidistant peaks at distance ΔS_1 is separated from the next group by the spacing ΔS_2, with $\Delta S_2 > \Delta S_1$. The number of peaks in a group is lower in the presence of quasiparticles in the bulk and when relaxation (fusion) between bulk and boundary neutral excitations is allowed.

The modulation follows from the energies of the neutral excitations, that are coupled to the charged ones by a $\mathbb{Z}_k$ selection rule, and from the non-Abelian fusion rules. These results led the authors of [3][4] to conclude that the pattern of Coulomb blockade peaks provide signatures for the non-Abelian statistics of the excitations. Analogous results were found in the case of the Ardonne-Schoutens non-Abelian spin singlet states [9].

In this paper, we discuss the case of the hierarchical Jain Hall states, with $\nu = m/(mp \pm 1)$, $m = 2, 3, \ldots$, $p = 2, 4, \ldots$, [10] that are described by the m-component Luttinger liquid and thus possess excitations with Abelian statistics [11]. We show that the same type of modulation in groups of m peaks occurs: however, it is independent of the presence of quasiparticles in the bulk and of bulk-edge relaxation phenomena (the latter are actually not possible). Furthermore, the peaks occur with a non-trivial pattern of multiplicities, owing to the extended symmetry of the multicomponent edge.

Our analysis shows that:

i) Although the modulation of conductance peaks is not by itself a characteristic feature of non-Abelian statistics of excitations, the complete pattern of peaks and its dependence on the bulk quasiparticles, provides an important experimental test of the CFT description, in particular of the fusion rules and other qualitative properties of the Hilbert space.

ii) The characteristic degeneracy of each conductance peak is an interesting signature of the composite edge structure: from its lifting, one can check additional interactions in the Hamiltonian, such as irrelevant terms, inter-edge couplings, edge reconstruction effects etc. (see e.g. Ref. [12]).

2. The partition function on the disk geometry

We start by describing the edge excitations by means of the CFT partition function on the disk geometry (droplets with different shapes are equivalent, owing to the symmetry of the QHE under area-preserving deformations [13]). The use of the (grand-canonical) partition function greatly simplifies the analysis: it not only accounts for all states in the theory but also for the selection rules which are built in. The partition functions for the prominent Hall states are already known in the annulus geometry [14][15], from which we shall deduce those for the case of the disk.

We illustrate the method and set the notation by first repeating the analysis of the simpler Laughlin states, $\nu = 1/q = 1, 1/3, 1/5, \ldots$. The CFT partition function describes excitations living on the two edges of the annulus and having opposite chiralities; the bulk is static and nothing depends on the radial coordinate. The angular and time coordinates are both periodic, the latter being the inverse temperature β: therefore, the spacetime geometry is that of a torus. The partition function is [14]:

$$Z = \sum_{\lambda=1}^{q} \left|\theta_\lambda(\tau,\zeta)\right|^2 \ , \tag{1}$$

in terms of the (extended) conformal characters,

$$\theta_\lambda(\tau,\zeta) = \mathrm{Tr}_{\mathcal{H}^{(\lambda)}}\left[\exp\left(i2\pi(\tau(L_0 - c/24) + \zeta Q)\right)\right] \ . \tag{2}$$

Their expressions can be exactly computed from the representation theory of the Virasoro algebra of conformal transformations and its extensions, including the current algebra (affine Lie algebra) $\widehat{U(1)}$. Each character resums the states on one edge of the annulus, within the Hilbert-space sector $\mathcal{H}^{(\lambda)}$ with charges $Q = \lambda/q + \mathbb{Z}$. The parameters in the exponent are given by $\mathrm{Im}\tau = v\beta/2\pi R$, in terms of the inverse temperature, Fermi velocity and edge length $2\pi R$, and by $2\pi\zeta = \beta(\mu + iV_o)$, in terms of the chemical and electric potentials, respectively. Therefore, states are collected according to their charge and energy: the latter is proportional to the conformal dimension, $E = (v/R)(L_0 - c/24)$, given by the zero mode L_0 of the Virasoro algebra, while the central charge c accounts for the Casimir energy [6].

In (1), the sectors relative to the chiral excitations on the outer edge of the annulus (accounted by θ_λ) are coupled to those of the inner edge ($\overline{\theta}_\lambda$) such that only integer charge excitations exist globally. In $\overline{\theta}_\lambda$, the variable $\overline{\tau}$ may be rescaled w.r.t. the complex conjugate of τ for allowing different parameters on the two edges: $\overline{v}/\overline{R} \neq v/R$. Furthermore, we shall extend τ to general complex values, $\mathrm{Im}\tau > 0$, such that the evolution in Z takes place both in time and space.

The annulus partition functions, such as (1), are given by bilinear combinations of characters that obey the conditions of invariance under modular transformations. These are coordinate transformations that respect the double periodicity of the torus [6]. The most interesting transformation exchanges the two periods, $S: \ Z(\tau,\zeta) = Z(-1/\tau, -\zeta/\tau)$. This is achieved in (1) by a linear transformation of the characters, $\theta_\alpha(-1/\tau) = \sum_\beta \ S_{\alpha\beta}\ \theta_\beta(\tau)$, with $S_{\alpha\beta}$ a unitary matrix. As discussed in Ref.[14], all modular conditions have physical interpretation in the Hall system. The S invariance amounts to a completeness condition for the spectrum of the theory and it is well known that the matrix $S_{\alpha\beta}$ also determines the fusion rules of the CFT [6]. A general result is that the number of sectors q in the partition function (1), i.e. the dimension of the S matrix, is equal to the topological order of the Hall state [5][14]. Two other modular conditions are, $T^2: \ Z(\tau+2,\zeta) = Z(\tau,\zeta)$ and $U: \ Z(\tau,\zeta+1) = Z(\tau,\zeta)$, that respectively impose fermionic statistics and integer charge for the excitations of the whole system on the two edges [14].

The disk partition function can be uniquely obtained from (1) by letting the inner radius $\overline{R} \to 0$, such that only the ground state contribution remains, $\overline{\theta}_\lambda \to \delta_{\lambda,0}$. Therefore, we find:

$$Z^{(0)}_{\rm Disk} = \theta_0 \ . \tag{3}$$

If, however, there are quasiparticles in the bulk of charge $Q_{\rm Bulk} = -\alpha/q$, then the condition of total integer charge selects another sector, $Z^{(\alpha)}_{\rm Disk} = \theta_\alpha$. In conclusion, the disk partition function is given by the chiral conformal character θ_α, with index selected by the bulk boundary conditions.

3. Coulomb blockade conductance peaks

We now proceed to compute the degeneracy of energy levels yielding the conductance peaks. For the Laughlin states, $\nu = 1/q$, the disk partition function is a sum of characters of the $\widehat{U(1)}$ affine current algebra with Virasoro central charge $c = 1$ and read, up to the constant $\exp\left(-\nu\pi({\rm Im}\zeta)^2/{\rm Im}\tau\right)$ [14]:

$$\theta_\lambda = K_\lambda(\tau,\zeta;q) \ , \tag{4}$$

with,

$$K_\lambda(\tau,\zeta;q) = \frac{1}{\eta}\sum_{n=-\infty}^{\infty} \exp i2\pi\left[\tau\frac{(nq+\lambda-\sigma)^2}{2q} + \zeta\frac{nq+\lambda}{q}\right] , \tag{5}$$

and $\eta(\tau)$ is the Dedekind function [6]. In this expression, n is the number of electrons added to the edge and $\sigma = B\Delta S/\phi_0$ is the deformation of the energy of edge excitations due to the variation ΔS of the area of the droplet. As explained in [3], this causes an unbalance of charge at the edge and an electrostatic energy cost.

On the other hand, for variations of the magnetic field B, the edge CFT is sensitive to the flux change, $\delta = \Delta\phi/\phi_o = S\Delta B/\phi_o$, as follows: the partition functions are deformed by $K_\lambda \to K_{\lambda+\delta}$, in (5) with $\sigma = 0$, i.e. the energies and charges of all states are changed. For one quantum of flux, $\delta = 1$, each λ-sector (θ_λ) goes into the following one: this is the so-called spectral flow [14]. For example, the ground state θ_0 becomes the one-anyon sector θ_1, meaning that the higher B has induced a quasi-hole of charge $Q = -1/q$ in the bulk and the edge states have adjusted correspondingly [16]. Note that the spectral flow leaves the annulus partition function (1) invariant; it amounts to the last modular invariance, $V: \ Z(\tau,\zeta+\tau) = Z(\tau,\zeta)$ [14]. The spectral flow encodes the Laughlin argument for the exactness of the Hall current [7]: after adding one quantum of flux, the system goes back to itself, but a charge equal to the filling fraction has moved from one edge to the other.

Consider the Hall dot without quasiparticles in the bulk corresponding to the partition function (3): from the character K_0 (5), we can extract the energies and charges of the electron excitations as the expressions multiplying τ and ζ, respectively. Upon deformation of the dot area, the ground state energy, $E \sim \sigma^2/2q$, $Q = 0$, and that of the one-electron state, $E \sim (\sigma - q)^2/2q$, $Q = 1$, become degenerate at $\sigma = q/2$. At this point one electron can tunnel into the dot causing a conductance peak. Similar degeneracies occur between consecutive multi-electron states. Therefore, the Coulomb blockade peaks are equally separated by the distance $\sigma = q$, corresponding to $\Delta S = \phi_o/(B\nu) = e/n_o$. In the presence of quasiparticles in the bulk with charge $Q = -\lambda/q$, one should repeat the analysis using the partition functions θ_λ (4): one obtains the same peak separations, because the energies in the different λ sectors are related by shifts of σ.

We have thus recovered the results of [3] from the study of the partition function: its decomposition into sectors makes it clear the allowed electron transitions. We conclude that the study of Coulomb blockade peaks provides insight into the qualitative and quantitative structure of the CFT Hilbert space of edge excitations.

4. Conductance peaks in hierarchical states

Let us now study the Coulomb blockade peaks in the hierarchical FQH states with filling fraction $\nu = m/(mp+1)$, $m = 2, 3, \ldots$, $p = 2, 4, \ldots$. The edge theory involves a m-component chiral Luttinger liquid, i.e. a $c = m$ CFT, with a symmetric spectrum of charges corresponding to the extended affine symmetry $\widehat{U(1)} \times \widehat{SU(m)}_1$ [11]. Our starting point is the annulus partition function, which is again of the general form (1), but where now each sector contains a non-trivial combination of $\widehat{U(1)}$ characters K_λ (5) and the characters $\chi_\alpha(\tau)$ for the neutral $(m-1)$ components [14]. The latter have no dependence on ζ and describe the spectrum with $\widehat{SU(m)}_1$ symmetry (in this special case, the affine symmetry actually leads to Abelian fractional statistics and fusion rules [6]). There are m neutral characters, obeying $\chi_{\alpha+m} = \chi_\alpha$; their leading low-energy behavior is, for $\mathrm{Im}\tau \to \infty$,

$$\chi_\alpha(\tau) \sim \binom{m}{\alpha} \exp i2\pi\tau \left(\frac{v_n}{v} \frac{\alpha(m-\alpha)}{2m} \right) + \cdots, \tag{6}$$

with $\alpha = 0, 1, \ldots, m-1$. In this expression, we allowed for a different velocity v_n of neutral excitations.

The topological order of hierarchical Hall fluids is given by the denominator of the filling fraction $q = mp + 1$ [11]; therefore, there are q terms in the annulus Z and correspondingly q possible disk partition functions $Z^{(a)}_{\text{Disk}} = \theta_a$, $a = 0, 1, \ldots, q-1$. Their expressions are [14]:

$$\theta_a(\tau, \zeta) = \sum_{\beta=1}^{m} K_{ma+\beta q}(\tau, m\zeta; mq)\ \chi_\beta(\tau)\ . \tag{7}$$

For example, in the $\nu = 2/5$ case, there are two neutral characters that combine with ten charged ones to obtain the following five sectors:

$$\begin{aligned} \theta_0\ &= K_0(\tau, 2\zeta; 10)\ \chi_0 + K_5(\tau, 2\zeta; 10)\ \chi_1\ , \\ \theta_{\pm 1} &= K_{\pm 2}(\tau, 2\zeta; 10)\ \chi_0 + K_{5\pm 2}(\tau, 2\zeta; 10)\ \chi_1\ , \\ \theta_{\pm 2} &= K_{\pm 4}(\tau, 2\zeta; 10)\ \chi_0 + K_{5\pm 4}(\tau, 2\zeta; 10)\ \chi_1\ . \end{aligned} \tag{8}$$

We now search for degeneracy of energy levels differing by the addition of one electron, $\Delta Q = 1$. Consider for definiteness the $\nu = 2/5$ case without any bulk quasiparticle, i.e. θ_0 above. From the expressions (5,7,8), one finds that the first term K_0 resumes all even integer charged excitations, while K_5 the odd integer ones. Therefore, the first conductance peak is found when the lowest energy state in $K_0\ \chi_0$, i.e. the ground state, with $E = (v_c/R)(2\sigma)^2/20$, $Q = 0$, becomes degenerate with the lowest one in $K_5\ \chi_1$, with $E = (v_c/R)(-5 + 2\sigma)^2/20 + v_n/4R$, $Q = 1$. The next peak occurs when the latter becomes degenerate with the first excited state ($Q = 2$) in $K_0\ \chi_0$, and so on. Note that in these energies, we changed $\sigma \to 2\sigma$, (cf. (5)), in order to respect the flux-charge relation, $\Delta Q = \nu\Delta\phi/\phi_o$. Owing to the contribution of the neutral energy in χ_1 (cf. (6)), the level matching is not midway and there is a bunching of peaks in pairs, with separations $\sigma = 5/2 \mp v_n/2v$.

For general m values, the result can be similarly obtained from (6,7). Our analysis is close to that of [3], because the neutral energies in (6) are equal, up to a factor of two, to those of the $\mathbb{Z}_m$ parafermions in the Read-Rezayi Halls states. For $\nu = m/q$, the resulting separation σ_k between the k-th and $(k+1)$-th peaks reads ($\sigma = B\Delta S/\phi_o$):

$$\begin{aligned}
\sigma_k &= \frac{q}{m} - \frac{v_n}{v}\frac{1}{m}\ , \qquad && d_k = \binom{m}{k}\ ,\ \ k = 1, \ldots, m-1, \\
\sigma_m &= \frac{q}{m} + \frac{v_n}{v}\frac{m-1}{m}\ , \qquad && d_m = 1\ .
\end{aligned} \tag{9}$$

The pattern repeats with periodicity m; in (9), we also report the degeneracy d_k for the k-th peak arising from the multiplicity factor in (6) (more on this later). Note that the result (9) could also be obtained from the analysis of the m-dimensional lattice of excitations [11], the multiplicity being given by the set of shortest vectors with integer charge k [17].

The formulae (9) also hold for the hierarchical states with $\nu = m/(mp - 1)$, upon replacing $q = mp - 1$: actually, the corresponding partition functions are of the form (7) with the replacement, $\chi_\beta \to \overline{\chi}_\beta$ [14], that does not affect the earlier discussion of energetics.

We thus found that the modulation of Coulomb blockade peaks is also possible in Abelian Hall states; the hierarchical states behave nevertheless differently from the non-Abelian Read-Rezayi states in the following aspects:

i) Most importantly, the neutral states (6) occur with a characteristic multiplicity d_k in (9), which is due to the $\widehat{SU(m)}_1$ symmetry. This means that several states become degenerate at the same point and d_k electrons tunnel into the dot simultaneously. Of course the CFT description applies to very large dots: in practice, this degeneracy is lifted by finite-size effect. Thus, the conductance peaks will have a comb-like substructure that could be observable by achieving higher experimental resolution.

ii) The pattern of peaks (9) is the same for any number of quasiparticles in the bulk, because the sectors, θ_a, $a \neq 0$, have linearly shifted energies w.r.t those of θ_0 (cf.(7)) [17]. Note, however, that bulk quasiparticles can have multiplicities, due to their neutral parts: e.g., for $\nu = 2/5$ there are two quasiparticles with $Q = 1/5$ (cf. (8)). When such multiplicity is $\binom{m}{k-1}$, the sequence of peaks (9) starts from σ_k (instead of σ_1) and goes on periodically.

iii) The relaxation processes between edge and bulk excitations are not possible. As explained in Ref.[4], the added electron at the boundary may decay into another excitation with same charge but different neutral content, by fusing with a neutral bulk quasiparticle. In the hierarchical states, any charged component, K_λ, appears only once in the spectrum (cf.(8)), thus relaxations cannot take place.

5. Discussion

The dynamics of the m-composite edge has been the subject of intense debate in the recent literature, starting from the experimental result [18]. Several deformations of, or additions to the Luttinger liquid Hamiltonian, have been put forward [12]; as these break the $\widehat{SU(m)}_1$ symmetry and possibly the conformal symmetry, they should lift the peak degeneracy.

Here we would like to recall the proposal of the minimal $W_{1+\infty}$ models [19]: in these theories, the degeneracy is completely eliminated and conformal symmetry is restored. As described in Ref.[20], a (non-local) projection in the edge Hamiltonian removes the $\widehat{SU(m)}_1$ symmetry while leaving the main feature of incompressibility of the Hall fluid, the so-called area-preserving of $W_{1+\infty}$ symmetry [13]. The conductance peaks in the minimal $W_{1+\infty}$ theories are easily obtained: since the projection preserves the structure of the Hilbert space of the CFT, it does not modify the earlier expressions for the partition functions, but only replaces the neutral characters by other expressions whose leading terms (6) do not contain the multiplicity factor [20]. We conclude that the observation of the conductance peaks without any multiplicity may provide a check for the $W_{1+\infty}$ minimal models. Further applications of disk partition functions will be given in a forthcoming publication [17].

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