Laws of Large Numbers, Spectral Translates and Sampling over LCA Groups.

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Abstract

Kluvánek extended the Whittaker-Kotel'nikov-Shannon theorem to the abstract harmonic analysis setting over a LCA group G. In this context, the classical condition for $f \in L^2(\mathbb{R})$ to be band limited is replaced by $\widehat{f}$ having its support essentially contained in a transversal set of a compact quotient group. This condition was later shown to be necessary in general. Moreover, the classical interpolation formula is also equivalent to a Plancherel like isometric formula involving the $L^2(G)$ norm of f and the norm of the sequence of its samples over a subgroup H. Here, recalling some Laws of Large Numbers, we will prove an equivalent result for the support of the spectral measure μ_X of a Gaussian stationary random process X, indexed over a LCA group G. The conditions are formulated in terms of an almost sure isometric formula involving the sample variances of X, and its samples over a subgroup H respectively.

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1 Introduction

A key result in Harmonic Analysis and Signal Processing over $\mathbb{R}^d$ is the so called *Whittaker-Kotel'nikov-Shannon* (WKS) sampling theorem which gives conditions to reconstruct (interpolate) a band limited $L^2(\mathbb{R}^d)$ function from its discrete samples taken at a uniform and appropriate rate. The WKS theorem was extended to $L^2(G)$, with G a *Locally Compact Abelian* (LCA) group, by Kluvánek [15]. In addition to its elegance, Kluvánek's result provides an example of a unified theory which gives a positive answer to several similar problems of uniform sampling that may seem different at first glance. In practice, if f represents a signal then its samples are obtained evaluating f over a subgroup $H \subset G$. Usually the case of interest is when H is countable. Kluvánek proved that a sufficient condition for the validity of a general interpolating formula, for $f \in L^2(G)$, is that the support of $\widehat{f}$ being essentially contained in a transversal set of $\widehat{G}/H^\perp$. Where $\widehat{G}$ is the dual group of G and $H^\perp$ the annihilator of H. Furthermore, in this case, an analogue of Plancherel's isometric formula holds for f and the sequence of its sampled values over H. For more details see Section 2.2.3. Converse results were presented in [2, 3].
Here, with the aid of certain Laws of Large Numbers (*LLNs* in short) for stationary random processes, we will replace the $L^2(G)$ norm by appropriate estimators of the variances of a stationary Gaussian random processes and its samples respectively. With this device we will prove that a similar *almost sure* isometric formula, in this case an analogue of Bohr's formula, is equivalent for a such a random process to have the support of the discrete part of its spectral measure contained in a transversal subset. Some of the ideas presented here, partially follow the techniques introduced in [6]. In that case, with the aid of some well known results form Ergodic Theory, it was proved that the equivalent and apparently unrelated notions

of *AP-Gabor frames* (see e.g. [4] and references therein) and $L^2(\mathbb{R})$-Gabor frames can be characterized in terms of an isometric formula involving the sample variance of a Gaussian stationary random process and the sample variance of its countable frame coefficients. The present work is tries to analyse further and bring more light to the relation between almost periodic functions, ergodic theorems and sampling. This is of certain importance in the sampling theory of finite-power signals, see e.g. [5].

Paper Organization.

The paper is organized as follows: Section 2 gathers most of the auxiliary results and previously known results, needed for our developments. Some of these results are originally spread separately in the literature. In order to make this work self-contained, some are presented as slight modifications of the original ones and a sketch of proof may be also given for a few of them. Most of the original results of this work are presented in Section 3. Theorems 6 and 7 relate conditions on the support of the spectral measure of the sampled random process X with an appropriate isometric formula. Finally, in Theorem 8 stability conditions are given.

2 Preliminaries

2.1 Some generalities.

If $(G,+)$ is a group, given $A \subseteq G$ and $B \subseteq G$ then we define the sum subset $A+B = \{x+y : \, x, y \in \, G\}$, and the subset of *differences* $A - B$ respectively. This must not be confused with the set theoretical difference $A \setminus B = A \cap B^c$. The symmetric difference is denoted by $A \Delta B$. Let U be a topological Hausdorff space and $\mathbf{B}(U)$ the Borel σ-algebra of U. By $\mathcal{M}(U)$ we will denote the *class of all complex-valued regular measures with finite total variation.* In the case that μ is a (real) signed measure associated the *variation* measure is given by $|\mu| = \mu_+ + \mu_-$. Where the pair μ_+, μ_- is given by the Hahn-Jordan decomposition of μ and the total variation norm is given by $\|\mu\|_{\mathcal{M}(U)} = |\mu|(U)$. So that $\mathcal{M}(U)$ is a normed linear space with $\|\,.\,\|_{\mathcal{M}(U)}$. Let W be a set and S any σ-algebra of W. If $\phi : U \longrightarrow W$ is a S-measurable map between U and W (i.e. a map such that $\phi^{-1}(S) \subseteq \mathbf{B}(U)$), we denote by $\phi^{-1}\mu$ the induced measure by ϕ, for $\mu \in \mathcal{M}(U)$. Following [20], a *support of* $\mu \in \mathcal{M}(U)$ is any $C \in \mathbf{B}(U)$ such that $|\mu|(C^c) = 0$. Alternatively, we may say that μ is concentrated on C, see e.g. p.266 of [30]. Note that this differs from the definition of support of a measure of p.124 of [13]. A measure μ is *discrete* if it is concentrated on a countable set; μ is *continuous* if $\mu(E) = 0$ for every countable set E. Every $\mu \in \mathcal{M}(U)$ has a unique decomposition $\mu = \mu_c + \mu_d$ where μ_c is continuous and μ_d is discrete. By $\mathcal{M}_d(U)$ we will denote the subset of discrete measures and by δ_x the unit mass measure concentrated on $x \in U$. The symbol $\mathbf{1}_S$ stands for the indicator function of the set S. The respective Lebesgue spaces of square integrable (equivalence classes) functions will be denoted by $L^2(U, \mathbf{B}(U), \mu)$ or $L^2(U, \mu)$ for short, if the underlying σ-algebra is clear from the context.

2.2 Fourier Transform and Function Spaces.

A LCA group G is a Hausdorff space with a locally compact topology which is an abelian group, provided that its group operation '$+$' (here written in additive form) is continuous. Denote by $\widehat{G}$ the dual group of G and by $\langle \gamma, x\rangle$ the value of $\gamma \in \widehat{G}$ at $x \in G$. If $H \subset G$ is a closed subgroup of G its annihilator is defined by

$$H^{\perp} = \{\gamma \in \widehat{G} : \langle x, \gamma\rangle = 1 \, , \forall \, x \in \, H\} \, .$$

Let m_G denote a fixed Haar measure on G, i.e. the unique, up to multiplicative positive constant, invariant measure with respect to the group operation '+'. For the case of the Haar measure m_G, the respective Lebesgue spaces of functions, for $p \in [1, \infty]$, are denoted by $L^p(G)$. Recall that in the case $p = 2$, the inner product is given by $\langle f, g\rangle = \int_G f(x)\overline{g(x)} dm_G(x)$. This notation should not be confused with the duality relation between G and $\widehat{G}$ although it will be clear form the context. If $f \in L^1(G)$, its Fourier Transform at

$\gamma \in \widehat{G}$ is defined by

$$\mathcal{F}f(\gamma) = \hat{f}(\gamma) = \int\limits_G f(t)\overline{\langle \gamma, t\rangle} dm_G(t)\,.$$

We will denote $A(\widehat{G}) := \mathcal{F}(L^1(G)) = \{\widehat{f} : \quad f \in L^1(G)\}$. On the other hand, the Haar measure $m_{\widehat{G}}$ over $(\widehat{G}, \mathbf{B}(\widehat{G}))$ can be adjusted so that the inverse Fourier transform for $\hat{f} \in L^1(\widehat{G})$ at $x \in G$ is given by: $f(x) = \int\limits_{\widehat{G}} \widehat{f}(\gamma)\langle \gamma, t\rangle dm_{\widehat{G}}(\gamma)\,.$ We need the following proposition. This is a variant of Theorem 1.6.4, p. 27, of [30], or alternatively this also a consequence of $A(\widehat{G})$ being a standard function algebra (Definition 2.1.1 of [27]):

Proposition 1 *If E is a nonempty open set of $\widehat{G}$ and $\gamma_0 \in E$, there exist a non-negative function $\widehat{f} \in A(\widehat{G})$, $\widehat{f}(\gamma_0) \neq 0$ such that $\widehat{f}(\gamma) = 0$ for all γ outside E.*

If $\mu \in \mathcal{M}(\widehat{G})$, we define its *inverse* Fourier Transform by:

$$\overset{\vee}{\mu}(t) = \int\limits_{\widehat{G}} \langle \gamma, t\rangle d\mu(\gamma)\,.$$

In fact, if μ is defined by $d\mu = f dm_{\widehat{G}}$ for some $f \in L^1(\widehat{G})$ then both definitions coincide. If $\mu \in \mathcal{M}(\widehat{G})$ it is uniquely determined by its Fourier transform and Bochner's theorem states that a continuous function on G is *positive-definite* if and only if is the Fourier transform of a non-negative measure $\mu \in \mathcal{M}(G)$. Obviously the roles of G of $\widehat{G}$ can be exchanged. However the present form seems to be more adequate for the presentation of subsequent results related to the theory of random processes.
A classical example of LCA group is $G = \mathbb{R}^d$ with its usual addition operation $+$, and its dual $\widehat{G} = \mathbb{R}^d$. In this case $\langle \gamma, x\rangle = e^{i\gamma \cdot x}$ with $dm_G(x) = \frac{dx}{(2\pi)^d}$ and $dm_{\widehat{G}}(\gamma) = d\gamma$, where dt and $d\gamma$ denote the usual Lebesgue measure of $\mathbb{R}^d$. Another usual example is the torus $G = \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and its dual $\widehat{G} = \mathbb{Z}$. In this case, $\langle \gamma, x\rangle = e^{i\gamma \cdot x}$ with $dm_G(x) = \frac{dx}{2\pi}$ and $m_{\widehat{G}}$ the counting measure. With these basic definitions in hand, the most relevant results for these classical cases, such as Plancherel's formula, can be extended with no difficulty to the general abstract setting of an LCA group G [13]. We shall need another class of functions [30]:

Definition 1 *The space $AP(G)$ is the uniform norm closure, in $C_B(G)$, of the space of trigonometric polynomials $p(t) = \sum\limits_{\gamma} C(\gamma)\langle \gamma, t\rangle$ with $\gamma \in \widehat{G}$ and $C(\gamma) \in \mathbb{C}$.*

Here $C_B(G)$ is the linear space of continuous and bounded functions over G. Also note that in Definition 1, with the abbreviated notation $p(t) = \sum\limits_{\gamma} C(\gamma)\langle \gamma, t\rangle$ and by a trigonometric polynomial we mean a finite linear combination of the form $p(t) = \sum\limits_{i=1}^{n} C(\gamma_i)\langle \gamma_i, t\rangle$ with $\gamma_i \in \widehat{G}$. In the forthcoming this will be clear from the context.

2.2.1 The Invariant Mean over G.

In this section we deal with the problem of defining a unique *invariant mean* over $AP(G)$ by an appropriate averaging process. A nice self-contained exposition of this topic is given in the article [23]. Here, the aim is to always keep the level of generality in proportion to our needs of showing a relation between LLNs and Sampling Theory. Recall that the Haar measure is unique up to scaling, so let us choose one such measure m_G to introduce the concepts of this section.

Definition 2 *A sequence $\mathcal{K} = \{K_n : n \in \mathbb{N}\}$ of non void, compact subsets of G is called a Følner sequence if the following conditions are satisfied:*

$$(i) \quad 0 < m_G(K_n), \;\; n \in \mathbb{N}, \qquad (ii) \;\; \frac{m_G((x + K_n)\Delta K_n)}{m_G(K_n)} \underset{n \longrightarrow \infty}{\longrightarrow} 0 \;\; \text{for all}\, x \in G. \tag{1}$$

Moreover, we shall assume that G contains a Følner sequence satisfying the additional *Shulman* condition [19]. There exists $C > 0$ such that for all n:

$$m_G\left(\bigcup_{k<n}(K_n - K_k)\right) \leq C\, m_G(K_n)\,. \tag{2}$$

Følner sequences allows us to define a *mean* for $f \in AP(G)$ as:

$$\mathbb{M}_{\mathbb{G}}^{\mathcal{K}}(f) = \lim_{n\longrightarrow\infty} \frac{1}{m_G(K_n)} \int\limits_{K_n} f(t) dm_G(t)\,. \tag{3}$$

If G is an LCA group, it follows that the value of $\mathbb{M}_{\mathbb{G}}^{\mathcal{K}}(f)$ is independent of the choice of the Følner sequence so, if $f \in AP(G)$, we can write $\mathbb{M}_{\mathbb{G}}(f)$. In fact there exists a unique *invariant mean* over $AP(G)$. See e.g. Theorem 18.10, p.252 of [13] or Section 4.5. of [23]. On the other hand, the additional condition of equation (2) may be crucial for the validity of the Pointwise Ergodic Theorem (in Section 2.3.7). For general topological groups, see e.g. [19] or p. 212 [24] and references therein for alternative conditions. So, in the forthcoming we will assume that G contains a Følner sequence verifying condition (2). This is not a great restriction. It is known that every σ-compact LCA group G admits a Følner sequence verifying condition (2). This is a consequence of Proposition 1.4 of [19].

2.2.2 Besicovitch Almost Periodic Functions.

The following inner product is well defined over $AP(G)$:

$$(f, g)_{AP(G)} = \lim_{n\longrightarrow\infty} \frac{1}{m_G(K_n)} \int\limits_{K_n} f(t)\overline{g(t)} dm_G(t) = \mathbb{M}_{\mathbb{G}}(f\overline{g})\,. \tag{4}$$

The norm induced by this inner product makes $AP(G)$ a non-complete inner-product space and, if for example $G = \mathbb{R}$, a non-separable space. For $\mathcal{K} = \{K_n : n \in \mathbb{N}\}$ a Følner sequence, we introduce the Hilbert space $B^2(G, \mathcal{K})$ of *Besicovitch almost periodic functions* containing $AP(G)$. Let $f \in L^2_{loc}(G)$ we can define the semi-norm:

$$\|f\|_{b,\,2,\,\mathcal{K}} = \left(\limsup_{n\longrightarrow\infty} \frac{1}{m_G(K_n)} \int\limits_{K_n} |f(t)|^2 dm_G(t)\right)^{\frac{1}{2}}.$$

A function $f \in L^2_{loc}(G)$ is called *Besicovitch* almost periodic, with respect to $\mathcal{K}$ if for every $\varepsilon > 0$ there exists $g \in AP(G)$ such that $\|f - g\|_{b,\,2,\,\mathcal{K}} < \varepsilon$. It is possible to turn $B^2(G, \mathcal{K})$ into a Hilbert space. First one can define an equivalence relation on $B^2(G, \mathcal{K})$ in the following way: $f \equiv g$ if and only if $\|f - g\|_{b,\,2,\,\mathcal{K}} = 0$. The norm of $[f] \in B^2(G, \mathcal{K})/\equiv$ is given by $\|[f]\|_{B^2(G,\mathcal{K})} := \|f\|_{b,2,\mathcal{K}}$. In particular if $f \in L^2_{loc}(G)$ is such that $\|f\|_{b,\,2,\,\mathcal{K}} = 0$ then $f \equiv 0$. Finally, one can prove that $B^2(G, \mathcal{K})/\equiv$ is complete (see e.g. p.39 of [18]). With some abuse, if there is no confusion, we will write f instead of $[f]$ the equivalence class of f and $B^2(G, \mathcal{K})$ for $B^2(G, \mathcal{K})/\equiv$. Moreover, it can be proved that the inner product $(\,.\,,\,.\,)_{B^2(G,\mathcal{K})}$ in $B^2(G, \mathcal{K})$ coincides with equation (4) for any $f, g \in B^2(G, \mathcal{K})$. We recall that $\{\langle\gamma, t\rangle : \gamma \in \widehat{G}\}$ forms a complete orthonormal basis of $B^2(G, \mathcal{K})$ and that the following analogue of Plancherel identity holds:

$$\|\, f \,\|_{B^2(G,\mathcal{K})} = \|\, C(f) \,\|_{L^2(\widehat{G},dc)}\ , \tag{5}$$

where $C(f)(\gamma) = (f, \langle\gamma, .\rangle)_{B^2(G,\mathcal{K})} = \lim_{n\longrightarrow\infty} \frac{1}{m_G(K_n)} \int\limits_{K_n} f(t)\overline{\langle\gamma, t\rangle} dm_G(t)$ denotes the *Fourier-Bohr coefficient* of f at $\gamma \in \hat{G}$ (or *Fourier-Bohr transform* in some literature) and c denotes the counting measure. Obviously, $C(f)(\gamma) = 0$ for all $\gamma \in \widehat{G}$ except for a finite or countable subset of them. We summarize some remarkable facts about are $B^2(G, \mathcal{K})$:

1. (Riesz-Fischer property) Let $C : \widehat{G} \longrightarrow \mathbb{C}$ and let $\mathcal{K}$ be a Følner sequence. Then there exists a unique $f \in B^2(G,\mathcal{K})$ (as an equivalence class) such that $C(\gamma) = (f, \langle\gamma, .\rangle)_{B^2(G,\mathcal{K})}$ if and only if $\| C \|_{L^2(\widehat{G},dc)} < \infty$. In this case $f = \sum_{\gamma\in\widehat{G}} C(\gamma)\langle\gamma, .\rangle$, where the convergence is in the $B^2(G,\mathcal{K})$-norm.

2. It follows that despite the spaces $B^2(G,\mathcal{K})$ may be different for distinct choices of $\mathcal{K}$, all of them are isometrically isomorphic.

3. Let $f \in AP(G)$, $\tau \in G$ and let $T_\tau f = f(\,.\,+\tau) \in AP(G)$ be the translation of f by τ. Then $\| f \|_{B^2(G,\mathcal{K})} = \| T_\tau f \|_{B^2(G,\mathcal{K})}$ and hence T_τ extends uniquely to an isometry on $B^2(G,\mathcal{K})$ (For more details see p. 39 of [18]). Here, with some abuse we will shall also denote this extension by T_τ. Moreover, one can define the *deterministic auto-correlation* of $f \in B^2(G,\mathcal{K})$ at $\tau \in G$ by:

$$\rho_f(\tau) := (\, f \,,\, T_\tau f \,)_{B^2(G,\mathcal{K})} \,. \tag{6}$$

If $f \in B^2(G,\mathcal{K})$ then $\rho_f \in AP(G)$ (see e.g. Lemma 2.1. of [17]). In fact, from (5) one can deduce that

$$\rho_f(\tau) = \sum_{\gamma\in\hat{G}} |C(\gamma)|^2\overline{\langle\tau,\gamma\rangle}\,,$$

and therefore ρ_f is the Fourier transform of the discrete measure $\nu = \sum_{\gamma\in A} |C(\gamma)|^2\delta_\gamma$.

For more details about these facts, see [23] or [13] for more general results. For the case $G = \mathbb{R}^d$ another interesting introductory article is [21].
Finally, if $H \subset G$ is a closed subgroup carrying a Haar measure m_H and containing a Følner sequence, a similar argument leads to existence of a Følner sequence $\mathcal{K}' = \{K'_n : n \in \mathbb{N}\}$ for H which verifies (2). Subsequently, one can define the mean $\mathbb{M}_H^{\mathcal{K}'}$ over H.

2.2.3 Paley-Wiener Spaces and Sampling in $L^2(G)$.

Let G be an LCA group. Analogously to the $\mathbb{R}^d$ case, for a measurable $S \subset \widehat{G}$ such that $m_{\widehat{G}}(S) < \infty$, one can define the Paley-Wiener spaces of S-band limited functions as: $PW_S = \{f \in L^2(G) : \operatorname{supp}(\hat{f}) \subseteq S\}$. In the context of sampling, the following definition is useful:

Definition 3 *[7] Let G be a LCA group. A discrete subgroup $H \subseteq G$ for which G/H is compact is called a lattice.*

Note that G/H being compact is equivalent to $H^\perp$ being discrete. In some literature, e.g. [8], the condition of H being countable is included in the definition of a lattice. Now, we can introduce Kluvanek's Sampling Theorem. Given a lattice H, we shall use Kluvanek's original normalization procedure for the Haar measures m_G and $m_{\widehat{G}}$: First, noting that $\hat{G}/H^\perp$ is compact, one fixes $m_{\hat{G}/H^\perp}$ so that $m_{\hat{G}/H^\perp}(\hat{G}/H^\perp) = 1$. Furthermore, let be $m_{H^\perp}(\{\lambda\}) = 1$ for all $\lambda \in H^\perp$ and $m_H(\{h\}) = 1$ for all $h \in H$. Once, $m_{\hat{G}/H^\perp}$ and $m_{H^\perp}$ are fixed, one can take $m_{\widehat{G}}$ so that Weil's formula holds, i.e. for every non-negative measurable f on $\widehat{G}$:

$$\int_{\widehat{G}} f(\gamma)dm_{\widehat{G}} = \int_{\widehat{G}/H^\perp}\int_{H^\perp} f(\gamma+\lambda)dm_{H^\perp}dm_{\widehat{G}/H^\perp}\,.$$

Finally, m_G is taken so that the Fourier inversion formula holds. Under the assumption that H is a lattice, Kluvanek's fundamental result for *error free* reconstruction of a PW_S-function from its samples in H is:

Theorem 1 *Let H be a lattice and let S be a Borel measurable subset of $\widehat{G}$ such that:*

$$S \cap S+\lambda = \emptyset \quad \textit{for all} \quad \lambda \in H^\perp \setminus \{0\}\,, \qquad \bigcup_{\lambda\in H^\perp} S+\lambda = \widehat{G}\,; \tag{7}$$

and let $f \in PW_S$. Then f is equal a.e. to a continuous function and if f is itself continuous then:

$$f(t) = \sum_{h \in H} f(h) k_S(t-h)\,, \tag{8}$$

where the convergence is uniform on $t \in G$ and in the $L^2(G)$-norm, and $k_S = \mathcal{F}^{-1}\mathbf{1}_S$. Moreover,

$$\|f\|^2_{L^2(G)} = \int\limits_G |f|^2 dm_G = \int\limits_H |f|^2 dm_H = \|f|_H\|^2_{L^2(H)}\,. \tag{9}$$

The translate condition $S \cap S + \lambda = \emptyset$ of Theorem 1 can be weakened to $m_{\widehat{G}}(S \cap S + \lambda) = 0$. Every subset S which verifies (7) it is said to be a *transversal* subset. Note that a transversal may not be a compact set. The classical example is the WSK Theorem, with $G = \widehat{G} = \mathbb{R}$, $H = t_0\mathbb{Z}$, $H^\perp = \frac{2\pi}{t_o}\mathbb{Z}$ and $T = [-\pi/t_0, \pi t_0)$. Equations (7) are *anti-aliasing* conditions. Moreover, in [1] for $G = \mathbb{R}$, it is proved that this last condition is also necessary for the validity of a perfect reconstruction formula as (8). Furthermore, it can be proved that condition (9) is equivalent to (8) [1, 2]. So these results can be stated departing from equation (9). In Theorem 9 of [3] an stability condition is given in terms of an open subset of $L^2(G) \bigcap C(G)$ for which formula (8) holds.
Note that if $f \in L^2(G) \bigcap B^2(G, \mathcal{K})$ then $f \equiv 0$ (as an element of $B^2(G, \mathcal{K})$). So in general, the tools developed for the $L^2(G)$ setting are not directly applicable to $B^2(G, \mathcal{K})$. This fact justify, in part, the present work. Note also that the original statements for $L^2(G)$ generally hold *a.e.* with respect to m_G (or $m_{\widehat{G}}$) the Haar measure associated to the separable space $L^2(G)$. In contrast the non separable $B^2(G, \mathcal{K})$ space, in some way, is associated to the counting measure. As a consequence, in the present case, the results are true for all the members of a certain subset and not only a.e..

2.3 Probability and Random Processes.

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space and X a random variable defined on it. If φ is any Borel measurable real or complex function, we denote $\mathbf{E}(\varphi(X))$ the expectation of $\varphi(X)$. The following brief description of the Theory of stationary random processes follows closely [25, 29]. Let $X = \{X(t),\ t \in G\} \subset L^2(\Omega, \mathcal{A}, \mathbf{P})$ be a real or complex, mean square continuous wide sense stationary (*w.s.s.* for short) random process, i.e. X verifies the following three conditions, for all $t, s \in G$:

$$(i)\ \mathbf{E}(X(t)) = 0,\quad (ii)\ \mathbf{E}(X(t)\overline{X(s)}) = R_X(t-s)\,,\quad (iii)\ R_X(t) \text{ is a continuous function of } t \in G. \tag{10}$$

For simplicity and with no loss of generality we imposed that X has a mean equal to zero. A stronger notion is (strict) stationarity. i.e. if the shifted families $X^T = \{X(t+T),\ t \in G\}$ have the same finite distributions as X for all $T \in G$. A strictly stationary process is w.s.s. but the converse is not always true. If X is real and Gaussian both notions are equivalent. In the complex case some additional care may be necessary, see Section 2.3.2.

2.3.1 Harmonic Analysis of Stationary Random Processes.

If X is a w.s.s. random process it is known by Bochner's Theorem that there exists a non-negative Borel measure measure $\mu_X \in \mathcal{M}(\widehat{G})$, *the spectral measure*, such that

$$R_X(t) = \overset{\vee}{\mu}_X(t) = \int\limits_{\widehat{G}} \langle t, \gamma \rangle d\mu_X(\gamma)\,.$$

Conversely, if μ_X is a finite Borel measure, there exists X a w.s.s random process with μ_X as its spectral measure. Morover, X can be defined as a Gaussian process and in the complex case it can be chosen such the next condition (13) holds, see e.g. p.147 of [29].

Defining the Hilbert space of random variables

$$\mathcal{H}(X) = \overline{span}\ X\ \subset L^2(\Omega, \mathcal{F}, \mathbf{P}),$$

then the mean square estimation theory for stationary sequences is mainly based on *Kolmogorov's isomorphism*:

$$I : L^2(\widehat{G}, \mathbf{B}(\widehat{G}), \mu_X) \longrightarrow \mathcal{H}(X) \tag{11}$$

given by the *stochastic integral*:

$$I(f) = \int_{\widehat{G}} f(\gamma) d\Phi_X(\gamma)\,,$$

where Φ_X is the (orthogonal) random measure associated to X. In fact, if A is a Borel subset then μ_X and Φ_X are related by the following formulas: $\mathbf{E}|I(\mathbf{1}_A)|^2 = \mathbf{E}|\Phi_X(A)|^2 = \mu_X(A)$ and $\mathbf{E}\left|\int_{\widehat{G}} f d\Phi_X\right|^2 = \int_{\widehat{G}} |f|^2 d\mu_X$. Moreover X has the following *spectral representation*:

$$X(t) = I(\langle t,\,.\,\rangle) = \int_{\widehat{G}} \langle t, \gamma\rangle d\Phi_X(\gamma)\,. \tag{12}$$

A detailed description of these representations can be found in [26]. The complex Gaussian case deserves a brief discussion:

2.3.2 Gaussian Complex Processes.

In general, if X is w.s.s. complex random process, we assume that $X(t) = X_1(t) + iX_2(t)$ for all $t \in G$. Where X_i, $i = 1, 2$ are two stationary (cross)correlated *real* w.s.s. stationary random processes. If X is *Gaussian* and *complex* we shall impose, in addition to (10), the condition:

$$\mathbf{E}(X(t)X(s)) = 0 \ \text{ for all } \ t, s\ \in G. \tag{13}$$

Gaussian complex random processes or vectors verifying condition (13) are said to be *circular*. This requirement is usual in signal theory and moreover it makes X retain most of the usual properties of real Gaussian processes [29]:

- If two random variables belonging to the closed linear span of X are uncorrelated then they are independent.
- X is completely determined by R_X and its mean, in our case $\mathbf{E}(X(t)) = 0$ for all t.
- Condition (13) is preserved by all linear operations on X. For more details see Lemma 10 in the Appendix.
- If X is w.s.s. random process then X is a stationary processes.

Throughout this work, to avoid repetitions, if X is a complex Gaussian and stationary random process, we shall assume that (13) holds. For a single complex Gaussian random variable Z with its mean equal to zero, this condition is equivalent to $\mathbf{E}(Z^2) = 0$.

2.3.3 An example of random measure.

An example of a Gaussian random measure is the following. Given $\mu \in \mathcal{M}(\widehat{G})$, choose any orthonormal basis of $L^2(\widehat{G}, \mu)$, $\{\varphi_n :\ n \in \mathbb{N}\}$ and $\{C(n) :\ n \in \mathbb{N}\}$ a sequence of independent random variables such that $C(n) \sim \mathcal{N}(0, 1)$. Then define for any $A \in \mathbf{B}(\widehat{G})$: $\Phi(A) = \sum\limits_n C(n)\langle\varphi_n, \mathbf{1}_A\rangle_{L^2(\widehat{G},\mu)}$. In particular, when $\mu = m_{\widehat{G}}$, Φ is the so called *Wiener measure*.

Following [29], linear time invariant filtering operations on X are defined, for any $t \in G$, by:

$$Y(t) = \int_{\widehat{G}} f(\gamma)\langle t, \gamma\rangle d\Phi_X(\lambda)\,,\ f \in L^2(\widehat{G}, \mu_X), \tag{14}$$

so the resulting stationary process $Y = \{Y(t)\}_{t\in G}$ can be thought as the output of a linear system with a frequency response given by f (i.e. *filter*) and a random input X. In this case, the covariance of Y is given by:

$$R_Y(t-u) = \mathbf{E}(Y(t)\overline{Y(u)}) = \int\limits_{\widehat{G}} |f(\gamma)|^2\langle t, \gamma\rangle d\mu_X(\gamma)\,. \tag{15}$$

Finally, the spectral measure μ_X can be decomposed into a continuous and purely discrete part $\mu_{X\,c}$ and $\mu_{X\,d}$ and there exists measurable and disjoint subsets C, D such that $\mu_{X\,c}(A) = \mu_X(A\cap C)$ and $\mu_{X\,d}(A) = \mu_X(A\cap D)$. From this we can give an orthogonal decomposition of X, $X(t) = \int_C\langle t, \gamma\rangle d\Phi_X(\gamma) + \int_D\langle t, \gamma\rangle d\Phi_X(\gamma) = X_c(t) + X_d(t)$ a.s.. In the Gaussian case X_c and X_d are independent. In the complex case, see Lemma 10 in the Appendix for a justification. This corresponds to the case when one replaces $f = \mathbf{1}_C$, or $f = \mathbf{1}_D$, in (14). For short, X_c and X_d will be called the *continuous* and *discrete* parts of X respectively. If μ_X is discrete then it is concentrated over a (countable) subset D_X of $\widehat{G}$ and moreover (12) takes the form of a random series:

$$X(t) = I(\langle t, \,.\,\rangle) = \sum_{\gamma\in D_X}\langle t, \gamma\rangle\Phi_X(\{\gamma\})\,. \tag{16}$$

In this case, we shall say that the process X has *discrete spectrum*, and in contrast if $\mu_X = \mu_{X\,c}$ we will say that X has *continuous spectrum*. In addition, if X is Gaussian and real (or complex verifying (13)), then the terms of the series (16) are independent random variables, in the complex case this is a consequence of e.g. Lemma 10 in the Appendix. A random process, for fixed $\omega \in \Omega$ may not be a measurable function of $t \in G$. To avoid pathological cases, condition (iii) gives a sufficient condition for the existence of a measurable process and equivalent to X. More generally, under rather mild conditions, if X is represented by a stochastic integral in practice it can be regarded as a measurable process, i.e. $X : \Omega \times G \longrightarrow \mathbb{C}$ is a measurable function with respect to the complete product σ-algebra $\widetilde{\sigma}(\mathbf{B}(G)\otimes\mathcal{F})$. This is the case of stationary processes. In fact, we have the following result which is an adaptation of one presented in [11] (For more details see the Appendix):

Lemma 1 *Let Φ be an orthogonal random measure over $(\widehat{G}, \mathbf{B}(\widehat{G}))$ with control measure $\mu \in \mathcal{M}(\widehat{G})$, and let $\nu \in \mathcal{M}(G)$ be such that ν is equivalent to m_G. If $\varphi \in L^2(G\times\widehat{G}, \mathbf{B}(G)\otimes\mathcal{F}, \nu\otimes\mu)$ and $\{X(t), t \in G\}$ is defined by*

$$X(t) = \int\limits_{\widehat{G}} \varphi(t, \gamma)d\Phi(\gamma)\,, \tag{17}$$

Then there exists a measurable process $\widetilde{X}$ stochastically equivalent to X. (i.e. $\mathbf{P}(\widetilde{X}(t) = X(t)) = 1$ for all $t \in G$)

In this Lemma by *control measure*, we mean the measure defined by $\mu(\,.\,) = \mathbf{E}|\Phi(\,.\,)|^2$. For example, if G is σ-finite, an equivalent measure $\nu \in \mathcal{M}(G)$ can be obtained in the following way: Since there exists countable disjoint subsets $K_n \in \mathbf{B}(G)$, $n \in \mathbb{N}$ such that $m_G(K_n) < \infty$ and $\bigcup_n K_n = G$, we can set $\nu(A) = \sum\limits_{n=1}^{\infty} 2^{-n}\frac{m_G(A\cap K_n)}{1+m_G(K_n)}$. In our context, the stationary case is obtained when $\varphi(t, \gamma) = \langle t, \gamma\rangle$. Lemma 1, in some sense, allows us to regard X as a measurable function. In view of this, in the forthcoming we shall assume this fact with no further mention of it when dealing with *"measurable"* operations over X. Finally, in order to make the presentation self contained we present an adaptation to the LCA group context, of a result of ([6] or [26] for an alternative argument) which will be useful in the sequel.

Lemma 2 *Let X be a wide sense stationary random process with associated* random *spectral measure* Φ_X *and let $f \in L^1(G)$. Then:*

$$\int_G X(t)\overline{f(t)}dm_G(t) = \int_{\widehat{G}} \overline{\widehat{f}(\gamma)}d\Phi_X(\gamma) \ \ a.s.$$

2.3.4 Example.

Convolution of a stationary random process X with a function $f \in L^1(G)$ (or $\hat{f} \in A(\widehat{G})$). This is a "time domain" interpretation of the filtering operation of equation (14). In fact, by Lemma 2, equation (14) can be rewritten as:

$$(X * f)(t) = \int_G X(s)f(t-s)dm_G(s) = \int_{\widehat{G}} \widehat{f}(\gamma)\langle t, \gamma\rangle d\Phi_X(\gamma) \ \ a.s. \tag{18}$$

In general, the stochastic integrals with respect to Φ_X are interpreted in the mean square sense (i.e. in the $L^2(\Omega, \mathcal{A}, \mathbf{P})$-norm).

2.3.5 Random Processes and Sampling.

There is not a complete analogue result to Theorem 1 for w.s.s. random processes involving the convergence of an interpolating series under a somewhat general condition as in the $L^2(G)$ case. However one can formulate a weaker result in terms of the support of the spectral measure μ_X and its translates (See next Theorem 2). Let $X = \{X(t),\ t \in G\}$ be a zero mean w.s.s. random process and let $H \subset G$ be a closed subgroup. If by $X|_H$ we denote the *sampled* random process $X|_H = \{X(t),\ t \in H\}$ and the canonical projection map by

$$\pi : \widehat{G} \underset{\gamma \mapsto [\gamma]}{\longrightarrow} \widehat{G}/H^\perp$$

then one can verify the following intuitive relation between the spectral measures of X and $X|_H$.

Lemma 3 *Let μ_X be the spectral measure of X. Then: $\pi^{-1}\mu_X \in \mathcal{M}(\widehat{G}/H^\perp)$ and $\mu_{X|_H} = \pi^{-1}\mu_X$.*

Proof. The regularity of $\pi^{-1}\mu_X$ follows from e.g. Lemma 2.1. of [22]. Let us obtain an expression of $\mu_{X|_H}$, the spectral measure of $X|_H$. Since the Fourier transform of this measure is the covariance function, for $t \in H$, of $X|_H$:

$$R_{X|_H}(t) = \int_{\widehat{G}/H^\perp} \langle t, [\gamma]\rangle d\mu_{X|_H}([\gamma]).$$

On the other hand, by the change of variable induced by the projection map $\pi : \widehat{G} \longrightarrow \widehat{G}/H^\perp$, a direct calculation for any $t \in H$ gives:

$$R_{X|_H}(t) = \mathbf{E}(X|_H(t)\overline{X|_H(0)}) = R_X(t) = \int_{\widehat{G}} \langle t, \gamma\rangle d\mu_X(\gamma) = \int_{\widehat{G}/H^\perp} \langle t, [\gamma]\rangle d\pi^{-1}\mu_X([\gamma])\,.$$

Then by the uniqueness of the Fourier transform we obtain $\mu_{X|_H} = \pi^{-1}\mu_X$. So that $\pi^{-1}\mu_X$ is the spectral measure of $X|_H$. □

We recall the following known and related result:

Theorem 2 *Let X be a wide sense stationary random process over a LCA group G, with spectral measure μ_X, and let $H \subset G$ be a closed subgroup with countable annihilator $H^\perp$. Then $\mathcal{H}(X) = \mathcal{H}(X|_H)$ if and only if there exists $S_X \in \mathbf{B}(\widehat{G})$ such that μ_X is concentrated in S_X and $S_X \bigcap S_X + \lambda = \emptyset$ for all $\lambda \in H^\perp \setminus \{0\}$.*

Although one may not have an interpolating series, the result gives a condition, of statistical value, under which X is completely and linearly determined by the samples $X|_H$. This theorem is originally formulated in terms of density conditions for trigonometric polynomials in e.g. [22]. (For more references on similar conditions see [16, 20]) However, in view of (11), the two statements are equivalent. In fact, the result holds for $L^\alpha(\widehat{G}, \mu)$ and the condition of $H^\perp$ being countable can be replaced by $\widehat{G}$ being a Polish space. As we will see, we cannot formulate this condition in terms of an isometric formula as in Theorem 1 for the $L^2(G)$ case, unless we restrict to the class of random processes with discrete spectrum. See Theorem 6 herein. In this context, we shall also formulate an stability condition analogue to Theorem 9 of [3], in terms of an open subset of $\mathcal{M}(\widehat{G})$. See Theorem 8 of section 3.1.1. Finally, observe that the condition of $H^\perp$ being countable in Theorem 2 is, in general, different from the condition in Theorem 1 of being a lattice. This is shown in the following example from Chapter 21 of [7]:

2.3.6 Example.

Let $G = \mathbb{R}^d$. In this case, the condition of H being a lattice is less general than $H^\perp$ being countable. In fact:

- Every closed subgroup H of $\mathbb{R}^d$ such that $H^\perp$ is countable has the form $H = A(\mathbb{R}^s \times \mathbb{Z}^{d-s})$ with $A \in \mathbb{R}^{d\times d}$ an invertible matrix and $s \in \{0, 1, \ldots, d\}$. In this case $H^\perp = (A^{-1})^T(\{0\}^s \times \mathbb{Z}^{d-s})$.
- Every lattice in $\mathbb{R}^d$ has the form $H = A\mathbb{Z}^d$ with $A \in \mathbb{R}^{d\times d}$ an invertible matrix. Then $H^\perp = (A^{-1})^T(\mathbb{Z}^d)$.

2.3.7 The Ergodic Theorems.

Natural estimators of the mean, variance and other statistics of a stationary process X are appropriate time averages. Convergence results for these averages take the form *Law of Large Numbers* for stationary processes or *Ergodic Theorems.* Here we gather some known results which will be useful in the sequel.
We say that the strictly stationary process X is *metrically transitive* (or equivalently *ergodic*) if the only measurable sets which are invariant under the *shift* $X \longmapsto X^T = \{X(t+T),\ t \in G\}$ have probability zero or one. Let G carry a Haar measure m_G, under the assumption that it contains a Følner sequence $\mathcal{K} = \{K_n : n \in \mathbb{N}\}$ verififying condition (2) E. Linderstrauss proved a version of the Pointwise Ergodic Theorem for general amenable groups [19]. We give an adaptation of this result which will be sufficient for our derivations.

Theorem 3 *Let X be a stationary random process and let $\mathcal{K}$ be a Følner sequence which verifies* (2). *Then:*

$$\mathbb{M}_G^{\mathcal{K}}(X) = \lim_{n\longrightarrow\infty} \frac{1}{m_G(K_n)} \int\limits_{K_n} X(t)dm_G(t)$$

exists a.s. and equals $\mathbf{E}(X(0)|\mathcal{F}_X)$. In particular, if the G action is ergodic, $\mathbb{M}_G^{\mathcal{K}}(X) = \mathbf{E}(X(0))$ a.s..

Here $\mathbf{E}(X(0)|\mathcal{F}_X)$ denotes the conditional expectation with respect the invariant sub σ-algebra $\mathcal{F}_X$. If $H \subseteq G$ is a closed subgroup, we can respectively define by an analogue averaging process the random variable denoted by $\mathbb{M}_H^{\mathcal{K}'}(X)$, for any random process indexed over H. One can deduce that X is metrically transitive if and only if in the above (a.s.) limit $\mathbb{M}_G^{\mathcal{K}}(X) = \mathbf{E}(X)$. Under the the weaker assumption on X of being a w.s.s. random process we have the following adaptation of the mean ergodic theorem. See for example [33] Theorem 2.1. p.481, or [9] , Theorem 1.

Theorem 4 *If X is a w.s.s. random process then, for every $\gamma \in \widehat{G}$, the limit*

$$\lim_{n\longrightarrow\infty} \frac{1}{m_G(K_n)} \int\limits_{K_n} \overline{\langle t, \gamma\rangle} X(t)dm_G(t) = \theta(\gamma)$$

exists in the mean square sense. Where $\theta(\gamma) \in H(X)$ is given by $\theta(\gamma) = \int_{\widehat{G}} \mathbf{1}_{\{-\gamma\}} d\Phi_X$. In particular, $\theta(\gamma) = 0$ a.s. if $\mu_X(\{-\gamma\}) = 0$.

Proof. (Sketch.) This is a consequence of Theorem 2.1. p.481 [33]. Define a new process $Y = \{Y(t) = \overline{\langle t, \gamma\rangle} X(t),\ t \in G\}$. It is easy to verify that Y is also a w.s.s. random process, and so that its spectral random measure is given by :

$$\Phi_Y(B) = \int_{\widehat{G}} \mathbf{1}_{B-\gamma} d\Phi_X \,.$$

In this case $\mathbb{M}_G^{\mathcal{K}}(Y)$ converges in the $L^2(\Omega, \mathcal{F}, \mathbf{P})$-norm to $\Phi_Y(\{0\}) = \Phi_X(\{-\gamma\})$.
In particular, $\mathbb{M}_G^{\mathcal{K}}(\overline{\langle t, \gamma\rangle} X(t)) = 0$ if $\mu_X(\{-\gamma\}) = 0$. □

For a general exposition on Ergodic Theorems for Group Actions see e.g. [31].

Remark.

Note that the last results can be adapted to the averages given by a.s. or *mean square sense* limits:

$$\mathbb{M}_G^{\mathcal{K}}(|X|^2) = \lim_{n \longrightarrow \infty} \frac{1}{m_G(K_n)} \int_{K_n} |X(t)|^2 dm_G(t) \ \text{ and } \ \rho_X(\tau) = \mathbb{M}_G^{\mathcal{K}}(X\overline{X(.+\tau)}) \ \text{ for any } \tau \in G \,.$$

Moreover, related to these we have the following useful Proposition. First, we recall the following formula. Let $(X_1, \ldots, X_4)$ be a Gaussian multivariate random vector then:

$$\mathbf{E}(X_i X_j X_k X_l) = \sigma_{i\,j}\sigma_{k\,l} + \sigma_{i\,k}\sigma_{j\,l} + \sigma_{i\,l}\sigma_{j\,k} \,, \tag{19}$$

where $\sigma_{i\,j}$ is the covariance of X_i and X_j. Note that a similar result holds in the complex case.

Proposition 2 *Let Let X be a zero mean Gaussian stationary random processes with continuous spectrum and let $\mathcal{K}$ be a Følner sequence which verifies* (2). *Then for all $\tau \in G$ the limit $\rho_X(\tau) = \mathbb{M}_G^{\mathcal{K}}(X\overline{X(.+\tau)})$ exists a.s. and equals $\mathbf{E}((0)\overline{X(\tau)}) = R_X(\tau)$.*

Proof. Define, for each $\tau \in G$, the stationary random process $Y_\tau = \{X(t)\overline{X(t+\tau)} - R_X(\tau),\ t \in G\}$. Let us obtain an xpression for its spectral measure.
First, assume that X is real, since X is Gaussian, recalling (19) we obtain the covariance function of Y_τ:

$$R_{Y_\tau}(t) = \mathbf{E}(Y_\tau(t))Y_\tau(0)) = \mathbf{E}(X(t)X(0)X(t+\tau)X(\tau)) - (R_X(\tau))^2 = (R_X(t))^2 + R_X(t-\tau)R_X(t+\tau) \,.$$

Define $d\mu_X^\tau(\gamma) = \langle t, \gamma\rangle d\mu_X$. Then:

$$R_X(t))^2 = \int_{\widehat{G}} \langle t, \gamma\rangle d(\mu_X * \mu_X)(\gamma) \,, \quad R_X(t-\tau)R_X(t+\tau) = \int_{\widehat{G}} \langle t, \gamma\rangle d(\mu_X^\tau * \mu_X^{-\tau})(\gamma) \,.$$

Therefore,

$$R_{Y_\tau}(t) = \int_{\widehat{G}} \langle t, \gamma\rangle d(\mu_X * \mu_X + \mu_X^\tau * \mu_X^{-\tau})(\gamma) \,,$$

and thus, since μ_X is continuous, the spectral measure of Y_τ, obtained by the sum of two convolutions, is also continuous. Consequently, by Theorem 4, $\mathbb{M}_G^{\mathcal{K}}(Y_\tau) = 0$ a.s. for each $\tau \in G$. Equivalently, by the definition of Y_τ, $\rho_X(\tau) = R_X(\tau)$ a.s., where the limit is taken in the $L^2(\Omega, \mathcal{F}, \mathbf{P})$-norm. However, by Theorem 3, the limit $\rho_X(\tau) = \mathbb{M}_G^{\mathcal{K}}(XX(.+\tau))$ exists a.s.. Thus, taking an appropriate sub-sequence $\{K_{n(k)} : k \in \mathbb{N}\}$, we get

$$\mathbb{M}_G^{\mathcal{K}}(XX(.+\tau)) = \lim_{k \longrightarrow \infty} \frac{1}{m_G(K_{n(k)})} \int_{K_{n(k)}} X(t)X(t+\tau) dm_G(t) = R_X(\tau) \ \textit{a.s.}$$

Therefore the limit exists a.s. and equals $R_X(\tau)$ as claimed.
The complex case is similar: Since X is Gaussian, recalling (19) and condition (13) we obtain the covariance function of Y_τ:

$$R_{Y_\tau}(t) = \mathbf{E}(Y_\tau(t))\overline{Y_\tau(0)}) = \mathbf{E}(X(t)\overline{X(0)}X(t+\tau)\overline{X(\tau)}) - |R_X(\tau)|^2 = |R_X(t)|^2.$$

But:

$$|R_X(t)|^2 = \int_{\widehat{G}} \langle t, \gamma\rangle d(\mu_X * \widetilde{\mu_X})(\gamma),$$

where $\widetilde{\mu_X})(A) = \overline{\mu_X(-A)}$ for all $A \in \mathbf{B}(\widehat{G})$. Therefore, Y_τ has a continuous spectral measure. Consequently, a similar argument to the real case gives that $\rho_X(\tau) = \mathbb{M}_G^{\mathcal{K}}(X\overline{X(.+\tau)}) = R_X(\tau)$ a.s. $\square$

Finally, note that in the particular case when $\tau = 0$ we get that $\mathbb{M}_G^{\mathcal{K}}(|X|^2) = \sigma_X^2$ a.s..

2.3.8 Random Series in $B^2(G,\mathcal{K})$.

Alternatively one may identify the whole trajectory of X as a random element of an appropriate Hilbert space $\mathcal{H}$. The following Theorem from [14], which is Hilbert space version of Kolmogorov's result for the sum of independent random variables, will be useful:

Theorem 5 *Let $\{X_n\,, n \in \mathbb{N}\}$ be a sequence of independent random elements in a Hilbert space $\mathcal{H}$ such that for all n: $\mathbf{E}(X_n) = 0$, and moreover $\sum_n \mathbf{E}\,\|X_n\|^2_{\mathcal{H}} < \infty$. Then $\sum_n X_n$ converges in $\mathcal{H}$ a.s..*

A direct application of this is the following:

Lemma 4 *Let X be a zero mean Gaussian stationary random process and let $\mathcal{K}$ be a Følner sequence in G which verifies* (2)*. Then:*
(i) If X has discrete spectrum then $X \in B^2(G,\mathcal{K})$ a.s.. Moreover, if (16) *is the spectral representation of X, then:* (16) *converges a.s. for all $t \in G$ and $\mathbb{M}_G^{\mathcal{K}}(|X|^2) = \|\, X \,\|_{B^2(G,\mathcal{K})} = \sum_{\gamma\in D_X} |\Phi_X(\{\gamma\})|^2$ a.s.*
(ii) Let G be a separable LCA group. If X has continuous spectrum and $X \in B^2(G,\mathcal{K})$ a.s. then X is the trivial null process. i.e. for every $t \in G$, $X(t) = 0$ a.s..

Proof. (i) Recall (16). If X has discrete spectrum, there exists $D_X \subset \widehat{G}$, a finite or countable subset such that the integral representation (12) takes the form of a random series:

$$X(t) = \int_{\widehat{G}} \langle t, \gamma\rangle d\Phi_X(\gamma) = \sum_{\gamma\in D_X} \langle t, \gamma\rangle \Phi_X(\{\gamma\})\,, \tag{20}$$

where the $\{\Phi_X(\{\gamma\})\}_{\gamma\in D_X}$ defines a sequence of zero mean independent Gaussian random variables. Note that,

$$\mathbf{E}\left|\sum_{\gamma\in D_X} \langle t, \gamma\rangle \Phi_X(\{\gamma\})\right|^2 = \sum_{\gamma\in D_X} \mathbf{E}|\Phi_X(\{\gamma\})|^2 = \sum_{\gamma\in D_X} \mu_X(\{\gamma\}) = \mu_X(\widehat{G}) < \infty\,,$$

since $\mu_X(D_X^c) = 0$. Then by the Three Series theorem $\sum_{\gamma\in D_X} \langle t, \gamma\rangle \Phi_X(\{\gamma\})$ converges a.s. for each $t \in G$. Moreover, since $\mathbf{E}(\langle t, \gamma\rangle \Phi_X(\{\gamma\})) = 0$ and $\sum_{\gamma\in D_X} \mathbf{E}\,\|\langle\,.\,, \gamma\rangle \Phi_X(\{\gamma\}\|^2_{B^2(G,\mathcal{K})} = \sum_{\gamma\in D_X} \mathbf{E}|\Phi_X(\{\gamma\})|^2 < \infty$, by Theorem 5, we get that the series representing X converges to an element of $B^2(G,\mathcal{K})$ a.s. with (random) Fourier coefficients given by $\Phi_X(\{\gamma\})$, $\gamma \in D_X$.

(ii) Recalling Property 3 of Section 2.2.2, since $X \in B^2(G,\mathcal{K})$ a.s. then $\mathbf{P}\,(\Omega_{AP}) = 1$ where

$$\Omega_{AP} = \{\omega \in \Omega:\ \rho_X(\,.\,,\omega) \in AP(G)\}\ .$$

Let D be a countable and dense subset of G and define, for each $d \in D$,

$$\Omega_d = \{\omega \in \Omega_{AP} : \ \rho_X(d,\omega) = R_X(d)\} \ \text{ and } \ \Omega_0 = \bigcap_{d \in D} \Omega_d \, .$$

Therefore, recalling Proposition 2, $\mathbf{P}(\Omega_0) = 1$. Noting that $\rho_X(d,\omega) = R_X(d)$ for all $d \in D$ and $\omega \in \Omega_0$ and that for each $\omega \in \Omega_0$: $\rho_X(\,.\,,\omega) - R_X \in C(G)$, then $\rho_X(t,\omega) = R_X(t)$ for all $t \in G$ and $\omega \in \Omega_0$. Now, suppose that $\sigma_X^2 = \mathbf{E}|X(0)|^2 = R_X(0) = \mu_X(\widehat{G}) > 0$. Then recalling again Property 3 of Section 2.2.2, for any $\omega \in \Omega_0$, $\rho_X(\,.\,,\omega)$ is the Fourier Transform of a non-zero discrete random measure. But, on the other hand R_X is the Fourier transform of a continuous measure, which is a contradiction by the uniqueness of the Fourier transform. $\square$

Brief Review and Remarks of Existing Related Results.

1. Observe that $X(t)$, for each $t \in G$, is an element of the Hilbert space $L^2(\Omega, \mathcal{F}, \mathbf{P})$. Consequently, many problems of this theory can be tackled using only the geometry of Hilbert spaces. This fact is exploited, by H. Feichtinger and W. Hörmann in e.g. [10] (and references therein). Let $\mathcal{H}$ be an arbitrary Hilbert space. In their context, an ordinary random process X can be seen as a measurable mapping $X : G \longrightarrow \mathcal{H}$. In this way, these random elements, can be treated as appropriate bounded linear mappings. this approach allows to avoid elegantly to a certain extent some technical results from Probability Theory. In the case of ordinary random processes, both definitions are compatible. However, the results presented herein rely on certain concepts of more measure theoretic nature (see e.g. Section 2.3.7), therefore a more classic approach may be desirable in the present case.

2. Alternatively, in Lemma 4, $\rho_X \in AP(G)$ a.s. since it is the Fourier transform of the measure $\sum\limits_{\gamma \in \widehat{G}} |\Phi_X(\{\gamma\})|^2 \delta_\gamma$, as a consequence of Lemma 4.8.10 of [23]. Furthermore, Lemma 4 seems to be related to some else of the results of Sec. 4.7 of [23] as was kindly suggested by one of the reviewers. However, note that in our case we are dealing with random objects and the statement (ii) $X(t) = 0$ a.s. for all $t \in G$ does not mean $X = 0$.

3 Spectral Translates and Sampling.

Kluvanek's Theorem relies mostly on Weil's and the Fourier inversion formulas, and as a consequence these impose some restrictions on the possible choices of the several Haar measures involved. In contrast, we shall rely mostly on the concepts introduced in Sections 2.2.1 and 2.2.2 where the main objective is to define an adequate mean over the group G (or H) and its subsequent consequences are, to a certain extent, independent of the particular normalization of the associated Haar measure.
To avoid repetitions in the statements of the theorems, the general assumptions made are the following:

- If X denotes a stationary random process , with no loss generality we assume that its mean is equal to zero.

- If X is a stationary Gaussian complex random process (or variable), we will assume that it verifies (13). Under this condition many proofs are almost the same as in the real case. So, unless otherwise stated, we will not distinguish between the two cases.

- G is a LCA group which carries a Haar measure. Again, since this is unique up to multiplicative constant, we shall choose one such measure, say m_G, once and for all the forthcoming. Additionally, we will suppose that G contains a Følner sequence $\mathcal{K} = \{K_n : n \in \mathbb{N}\}$ verifying condition (2).

- $H \subset G$ will be a closed subgroup containing also a Følner sequence $\mathcal{K}'$ which verifies (2) and a corresponding Haar measure m_H. So, we can think that we are working with a given pair of sequences $\mathcal{K}$ and $\mathcal{K}'$. These conditions can be achieved if, for example, G is σ-compact.

- H is such that its annihilator $H^{\perp}$ is countable (and therefore discrete).

Recall that there exists a Borel measurable transversal set T of the quotient group $\widehat{G}/H^{\perp}$ (see [2] and references therein). Note that several of the following results are stated in terms of the norm $B^2(G,\mathcal{K})$ (or $B^2(H,\mathcal{K}')$ respectively) as well as the means $\mathbb{M}_G^{\mathcal{K}}(\,.\,)$. These spaces and values are determined by the particular choice of $\mathcal{K}$ and $\mathcal{K}'$, but in contrast our claims and proofs are independent of the particular choice of $\mathcal{K}$ and $\mathcal{K}'$. This is a consequence of the following facts. Every Gaussian (real or complex under condition (13)) stationary random process can be decomposed in two independent parts X_c and X_d with continuous and discrete spectrum respectively. By Lemma 4 (i), X_d belongs to $B^2(G,\mathcal{K})$ a.s. and the (a.s.) value of its norm is independent of the sequence $\mathcal{K}$. On the other hand, we shall apply to X_c Theorems 3 and 4 which, in some sense, are also independent of $\mathcal{K}$ (and the particular choice of m_G).
It is clear that, for each realization of X, $X|_H$ can be seen as the restriction of X over H. First, let us study which information of X can be recovered from $X|_H$ and how are related their respective spectral structure. We begin with a rather intuitive result.

Lemma 5 *Let $X = \{X(t),\ t \in G\}$ be a w.s.s. stationary random process Then:*
$(X|_H)_d(t) = X_d|_H(t)$ and $(X|_H)_c(t) = X_c|_H(t)$ a.s. , for all $t \in H$.

Proof. Let μ_X be the spectral measure of X and let $D, C \in \mathbf{B}(\widehat{G})$, $D \cap C = \emptyset$ be such that $\mu_{X\,d}(A) = \mu_X(A \bigcap D)$ and $\mu_{X\,c}(A) = \mu_X(A \bigcap C)$ for all $A \in \mathbf{B}(\widehat{G})$. Then the spectral representation of X is given, for every $t \in H$, by:

$$X(t) = X_c(t) + X_d(t) = \int_{\widehat{G}} \langle t,\gamma\rangle \mathbf{1}_C(\gamma) d\Phi_X(\gamma) + \int_{\widehat{G}} \langle t,\gamma\rangle \mathbf{1}_D(\gamma) d\Phi_X(\gamma)\,.$$

Applying Lemma 3 to $X_d|_H$ we get that the spectral measure of $X_d|_H$ is given by $\pi^{-1}\mu_{X\,d} \in \mathcal{M}(\widehat{G}/H^{\perp})$. Noting that $\mu_{X\,d} = \sum\limits_{\gamma \in D} c_\gamma \delta_\gamma$ for some positive c_γ. Then for any $A \in \mathbf{B}(\widehat{G}/H^{\perp})$:

$$\pi^{-1}\mu_{X\,d}(A) = \sum_{\gamma \in D} c_\gamma \delta_\gamma(\pi^{-1}(A)) = \sum_{[\gamma] \in \pi(D)} c_\gamma \delta_{[\gamma]}(A)\,,$$

since $\delta_\gamma(\pi^{-1}(A)) = 1$ if and only if $\gamma \in \pi^{-1}(A)$, or equivalently if and only if $[\gamma] \in A$. Therefore $X_d|_H$ has discrete spectrum.
By a similar argument, one can obtain the following expression for $X_c|_H$:

$$\mu_{X_c|_H} = \pi^{-1}\mu_{X\,c}\,.$$

Now, we must check that $\pi^{-1}\mu_{X\,c}$ is a continuous measure. For $\gamma \in \widehat{G}$ its corresponding coset in $\widehat{G}/H^{\perp}$ is given by $[\gamma] = \gamma + H^{\perp}$. Then, recalling that $H^{\perp}$ is countable:

$$\pi^{-1}\mu_{X\,c}(\{[\gamma]\}) = \mu_{X\,c}(\gamma + H^{\perp}) = \sum_{\lambda \in H^{\perp}} \mu_{X\,c}(\{\gamma + \lambda\}) = 0\,.$$

Clearly $X_c|_H$ has continuous spectrum. The claim is proved since we have decomposed $X|_H$ in the sum of two processes, one with continuous spectrum: $X_c|_H$, and other $X_d|_H$, with discrete spectrum. $\square$

If X has discrete spectrum we have the following isometric formula.

Lemma 6 *Let $X = \{X(t),\ t \in G\}$ be a Gaussian stationary random process with discrete spectrum. Then: there exists $D_X \in \mathbf{B}(\widehat{G})$ a support of μ_X such that $D_X \bigcap (D_X + \lambda) = \emptyset$ for all $\lambda \in H^{\perp} \setminus \{0\}$, if and only if*

$$\|X\|^2_{B^2(G,\mathcal{K})} = \|X|_H\|^2_{B^2(H,\mathcal{K}')} \ \text{a.s.} \tag{21}$$

Proof. If X has discrete spectrum then it has the following representation for some countable $D_X \in \mathbf{B}(\widehat{G})$:

$$X(t) = \int_{\widehat{G}} \langle t, \gamma \rangle d\Phi_X(\gamma) = \sum_{\gamma \in D_X} \langle t, \gamma \rangle \Phi_X(\{\gamma\}) \,, \tag{22}$$

where $\mathbf{E}|\Phi_X(\gamma)|^2 = \mu_X(\{\gamma\}) > 0$ for all $\gamma \in D_X$ and $\mu_X(\{\gamma\}) = 0$ for all other γ. Recalling Lemma 4, $X \in B^2(G, \mathcal{K})$ a.s. and thus recalling equation (5):

$$\|X\|^2_{B^2(G,\mathcal{K})} = \sum_{\gamma \in D_X} |\Phi_X(\{\gamma\})|^2 \; a.s.. \tag{23}$$

On the other hand, by Lemma 5, $X|_H$ also has discrete spectrum and then, by Lemma 4, $X|_H \in B^2(H, \mathcal{K}')$ a.s.. We can find the *random* Fourier coefficients of $X|_H$ from $\Phi_X(\{\gamma\})$, $\gamma \in D_X$. In fact, for any $t \in H$, it is possible to rearrange the sum (22) in the following way:

$$X|_H(t) = X(t) = \sum_{\lambda \in H^\perp} \sum_{\gamma \in D_X \bigcap (T+\lambda)} \langle t, \gamma \rangle \Phi_X(\{\gamma\}) \,, \tag{24}$$

since $\langle t, \gamma \rangle = 1$ for all $t \in H$ and $\lambda \in H^\perp$. For each λ, one can make the change of variable $\gamma' = \gamma - \lambda$, so that $\gamma \in D_X \bigcap (T + \lambda)$ if and only if $\gamma' \in (D_X - \lambda) \bigcap T$. Therefore equation (24) takes the form:

$$= \sum_{\lambda \in H^\perp} \sum_{\gamma' \in (D_X - \lambda) \cap T} \langle t, \gamma \rangle \Phi_X(\{\gamma' + \lambda\}) = \sum_{\gamma' \in T} \left(\sum_{\lambda \in H^\perp} \mathbf{1}_{D_X - \lambda}(\gamma') \Phi_X(\{\gamma' + \lambda\}) \right) \langle t, \gamma' \rangle = \sum_{\gamma' \in T} \widetilde{\Phi}_X(\{\gamma'\}) \langle t, \gamma' \rangle \,.$$

So, if $\widetilde{\Phi}_X(\gamma') = \sum_{\lambda \in H^\perp} \mathbf{1}_{D_X - \lambda}(\gamma') \Phi_X(\{\gamma' + \lambda\})$, then again by equation (5) applied to H and $\widehat{H}$:

$$\|X|_H\|^2_{B^2(H,\mathcal{K}')} = \sum_{\gamma \in T} |\, \widetilde{\Phi}_X(\{\gamma\})|^2 \; a.s., \tag{25}$$

since $\widehat{G}/H^\perp \cong \widehat{H}$ (see e.g. p. 136 of [27]) and there exists a bijective correspondence with T.
Note that if $\Omega_1 = \{\omega \in \Omega : \|X(\omega)\|_{B^2(G,\mathcal{K})} < \infty\}$ and $\Omega_2 = \{\omega \in \Omega : \|X|_H(\omega)\|_{B^2(H,\mathcal{K}')} < \infty\}$ then, by Lemma 4, applied to X and $X|_H$ respectively, we obtain $\mathbf{P}(\Omega_1 \bigcap \Omega_2) = 1$. Additionally, we have that over $\Omega_1 \bigcap \Omega_2$, equations (23) and (25) are equal if and only if

$$\sum_{\gamma \in T} |\, \widetilde{\Phi}_X(\{\gamma\})|^2 = \sum_{\gamma \in D_X} |\, \Phi_X(\{\gamma\})|^2 \,. \tag{26}$$

But, $\sum_{\gamma \in T} |\, \widetilde{\Phi}_X(\{\gamma\})|^2$ is equal to:

$$\sum_{\gamma \in T} \left(\sum_{\lambda \in H^\perp} \mathbf{1}_{D_X}(\gamma + \lambda) |\, \Phi_X(\{\gamma + \lambda\})|^2 + \sum_{\lambda \neq \lambda'} \mathbf{1}_{(D_X - \lambda) \bigcap (D_X - \lambda')}(\gamma) \, \Phi_X(\{\gamma + \lambda\}) \overline{\Phi_X(\{\gamma + \lambda'\})} \right)$$

$$= \sum_{\gamma \in D_X} |\, \Phi_X(\{\gamma\})|^2 + \sum_{\gamma \in T} \sum_{\lambda \neq \lambda'} \mathbf{1}_{(D_X - \lambda) \bigcap (D_X - \lambda')}(\gamma) \, \Phi_X(\{\gamma + \lambda\}) \overline{\Phi_X(\{\gamma + \lambda'\})} \,. \tag{27}$$

Consequently, (26) holds if and only if

$$\sum_{\gamma \in T} \sum_{\lambda \neq \lambda'} \mathbf{1}_{(D_X - \lambda) \bigcap (D_X - \lambda')}(\gamma) \, \Phi_X(\{\gamma + \lambda\}) \overline{\Phi_X(\{\gamma + \lambda'\})} = 0 \quad a.s.$$

Taking the variance to this expression, this is equivalent to the condition:

$$\sum_{\gamma \in T} \sum_{\lambda \neq \lambda'} \mathbf{1}_{(D_X - \lambda) \bigcap (D_X - \lambda')}(\gamma)\, \mu_X(\{\gamma + \lambda\}) \mu_X(\{\gamma + \lambda'\}) = 0\,,$$

and recalling that $\mu_X(\{\gamma\}) > 0$ for all $\gamma \in D_X$, one gets that $\mathbf{1}_{(D_X-\lambda)\bigcap(D_X-\lambda')}(\gamma) = 0$ for all $\gamma \in T$ or equivalently $(D_X + \lambda) \bigcap D_X = \emptyset$ for all $\lambda \in H^\perp \setminus \{0\}$. $\square$

Note that if X has discrete spectrum then D_X is at most countable. If $A \in \mathbf{B}(\widehat{G})$ is arbitrary then we have the following derivation.

Corollary 1 *A subset $A \in \mathbf{B}(\widehat{G})$ verifies $A \bigcap (A + \lambda) = \emptyset$ for all $\lambda \in H^\perp \setminus \{0\}$ if and only if (21) holds for every Gaussian stationary process X with discrete spectrum such that $\mu_X(A^c) = 0$.*

Proof. The *only if* part of the claim is an immediate consequence of the previous Lemma. For the *if* part, suppose that for some $\gamma \in A$ and some $\lambda \in H^\perp \setminus \{0\}$: $\gamma + \lambda \in A$. Let $C(\gamma)$, $C(\gamma + \lambda)$ be two i.d.d's, zero mean Gaussian complex random variables. Recall that they can be chosen such that $\mathbf{E}(C(\gamma + \lambda)^2) = \mathbf{E}(C(\gamma)^2) = 0$. Then consider the auxiliary Gaussian stationary random process defined by $X(t) = C(\gamma)\langle t, \gamma\rangle + C(\gamma + \lambda)\langle t, \gamma + \lambda\rangle$ for every $t \in G$. Thus the sampled process $X|_H$ is given by $X|_H(h) = \langle h, \gamma\rangle(C(\gamma) + C(\gamma + \lambda))$ for every $h \in H$. Therefore, a direct calculation gives that (21) holds if and only if $\mathbf{P}(\mathbb{R}e(C(\gamma)C(\gamma + \lambda)) = 0) = 1$ but this is impossible since the joint probability distribution of $C(\gamma)$ and $C(\gamma + \lambda)$ is absolutely continuous with respect to the Lebesgue measure. $\square$

In Theorem 8 of the next section, one can see a kind of refinement to this last result. In contrast to Lemma 6, if X has continuous spectrum we have the following result.

Lemma 7 *If X is a Gaussian stationary process with continuous spectrum, then:*

$$\mathbb{M}_G^{\mathcal{K}}(|X|^2) = \mathbb{M}_H^{\mathcal{K}'}(|X|_H|^2) \quad a.s.$$

Proof. By Theorem 3, both limits $\mathbb{M}_G^{\mathcal{K}}(|X|^2)$ and $\mathbb{M}_H^{\mathcal{K}'}(|X|_H|^2)$ exist a.s.. We shall prove that these limits are equal. This will be a consequence of Proposition 2 with $\tau = 0$. In fact, $\mathbb{M}_G^{\mathcal{K}}(|X|^2) = \mathbf{E}(|X(0)|^2) = \sigma_X^2$ a.s..
On the other hand, from Lemma 5, one gets that $\mu_{X|_H}$ is continuous and then we can also apply Proposition 2 to $X|_H$, obtaining that $\mathbb{M}_H^{\mathcal{K}'}(|X|_H|^2) = \mathbf{E}(|X|_H(0)|^2) = \sigma_X^2$ a.s., which proves the claim.

$\square$

If, X_d and X_c denote the discrete and continuous parts of X respectively, we have the following (a.s.) orthogonality relation.

Lemma 8 *Let X be a Gaussian stationary random process. For $t \in G$ define the random process $Y(t) = X_c(t)\overline{X_d(t)}$. Then $Y = \{Y(t),\ t \in G\}$ is a stationary zero mean random process with continuous spectrum. Moreover, we have:*

$$\lim_{n \longrightarrow \infty} \frac{1}{m_G(K_n)} \int_{K_n} Y(t)\, dm_G(t) = \mathbb{M}_G^{\mathcal{K}}(Y) = \mathbb{M}_G^{\mathcal{K}}(X_c\overline{X_d}) = 0 \ \ a.s. \tag{28}$$

Proof. If X is stationary then the same holds for Y, since $Y(t)$ is defined as the composition of a measurable function with $X(t)$. Therefore, by Theorem 3, the limit (28) exists and is finite a.s.. On the other hand, recalling that X_c and X_d are independent, an easy calculation gives that $\mathbf{E}(Y(t)) = 0$. We can also calculate its covariance:

$$R_Y(t) = \mathbf{E}(Y(t)\overline{Y(0)}) = \mathbf{E}(X_c(t)\overline{X_c(0)}X_d(t)\overline{X_d(0)}) = \mathbf{E}(X_c(t)\overline{X_c(0)})\mathbf{E}(X_d(t)\overline{X_d(0)})$$

$$= R_{X_c}(t)R_{X_d}(t) = \int\limits_{\widehat{G}} \langle t, \gamma \rangle d(\mu_{X\,c} * \mu_{X\,d})(\gamma)\,.$$

Where the last equality is due to the convolution property of the Fourier Transform of measures. Then $\mu_{X\,c} * \mu_{X\,d}$ is the spectral measure of Y. But $\mu_{X\,c} * \mu_{X\,d}$ is continuous. In fact, if D_X is the (countable) support of $\mu_{X\,d}$, then for any $\gamma \in \widehat{G}$: $\mu_{X\,c} * \mu_{X\,d}(\{\gamma\}) = \sum\limits_{\gamma' \in D} \mu_{X\,c}(\{\gamma + \gamma'\})\mu_{X\,d}(\{\gamma'\}) = 0$ since $\mu_{X\,c}(\{\gamma\}) = 0$ for any γ. Thus, by Theorem 4, we get that:

$$\lim_{n \longrightarrow \infty} \frac{1}{m_G(K_n)} \int\limits_{K_n} Y(t) dm_G(t) = \mathbf{E}(Y(0)) = 0 \;\; a.s.$$

$\square$

A direct consequence is the following:

Lemma 9 *Let X be a Gaussian stationary random process, then:*

$$\mathbb{M}_G^{\mathcal{K}}(|X|^2) = \mathbb{M}_G^{\mathcal{K}}(|X_c|^2) + \mathbb{M}_G^{\mathcal{K}}(|X_d|^2) \;\; a.s. \tag{29}$$

Proof. Given $K_n \in \mathcal{K}$,

$$\frac{1}{m_G(K_n)} \int\limits_{K_n} |X(t)|^2 dm_G(t)$$

$$= \frac{1}{m_G(K_n)} \int\limits_{K_n} |X_c(t)|^2 dm_G(t) + \frac{1}{m_G(K_n)} \int\limits_{K_n} 2\mathbb{R}e(X_c(t)\overline{X_d(t)}) dm_G(t) + \frac{1}{m_G(K_n)} \int\limits_{K_n} |X_d(t)|^2 dm_G(t)$$

Define the events:
$\Omega_0 = \{\omega \in \Omega : \mathbb{M}_G^{\mathcal{K}}(|X|^2)(\omega) < \infty\}$, $\Omega_1 = \{\omega \in \Omega : \mathbb{M}_G^{\mathcal{K}}(|X_c|^2)(\omega) < \infty\}$, $\Omega_2 = \{\omega \in \Omega :\; \mathbb{M}_G^{\mathcal{K}}(|X_d|^2)(\omega) < \infty\}$ and

$$\Omega_3 = \left\{\omega \in \Omega :\; \mathbb{M}_G^{\mathcal{K}}(X_c \overline{X}_d) = 0\right\}\;.$$

Then, by Theorem 3, $\mathbf{P}(\Omega_i) = 1,\; i = 0, 1, 2$. In addition $\mathbf{P}(\Omega_3) = 1$ by Lemma 8, equation (28). Consequently, if $\widetilde{\Omega} = \cap_{i=0}^3 \Omega_i$, $\mathbf{P}(\widetilde{\Omega}) = 1$ then for every $\omega \in \widetilde{\Omega}$:

$$\mathbb{M}_G^{\mathcal{K}}(|X|^2) = \lim_{n \longrightarrow \infty} \frac{1}{m_G(K_n)} \int\limits_{K_n} |X(t,\omega)|^2 dm_G(t) =$$

$$\lim_{n \longrightarrow \infty} \left(\frac{1}{m_G(K_n)} \int\limits_{K_n} |X_c(t)|^2 dm_G(t) + \frac{1}{m_G(K_n)} \int\limits_{K_n} |X_d(t)|^2 dm_G(t) \right)\;,$$

which proves (29). $\square$

3.1 Main Results.

In Lemmas 6 and 9 we construct a subset-event of probability 1, where several required properties hold for a certain random process. It will be clear that in some of the reminding results one may repeat a similar argument. However, we left these details for the reader since it is assumed, at this point, that this is a clear exercise.

Theorem 6 *Let X be a stationary Gaussian random process. Then: there exists a support of $\mu_{X\,d}$, $D_X \in \mathbf{B}(\widehat{G})$, such that $D_X \bigcap D_X + \lambda = \emptyset$ for all $\lambda \in H^{\perp} \setminus \{0\}$, if and only if*

$$\mathbb{M}_G^{\mathcal{K}}(|X|^2) = \mathbb{M}_H^{\mathcal{K}'}(|X|_H|^2)\; a.s.$$

Proof. By Lemma 9 applied to $X|_H$, since the same result can be applied over H in the place of G, in addition to equation (29) we obtain:

$$\mathbb{M}_H^{\mathcal{K}'}(|X|_H|^2) = \mathbb{M}_H^{\mathcal{K}'}(|(X|_H)_c|^2) + \mathbb{M}_H^{\mathcal{K}'}(|(X|_H)_d|^2) \quad a.s. \tag{30}$$

$$= \mathbb{M}_H^{\mathcal{K}'}(|X|_H|^2) = \mathbb{M}_H^{\mathcal{K}'}(|X_c|_H|^2) + \mathbb{M}_H^{\mathcal{K}'}(|X_d|_H|^2) \quad a.s., \tag{31}$$

since by Lemma 5, $X_c|_H = (X|_H)_c$ and $X_d|_H = (X|_H)_d$ a.s..
By Theorem 3, the $\overline{\lim}$ can be replaced by a lim in

$$\|X_d\|^2_{B^2(G,\mathcal{K})} = \lim_{n\longrightarrow\infty} \frac{1}{m_G(K_n)} \int_{K_n} |X_d(t)|^2 dm_G(t) = \mathbb{M}_G^{\mathcal{K}}(|X|^2) \; a.s.$$

The same argument applies to $X|_H$. So that equation (31) becomes:

$$\mathbb{M}_H^{\mathcal{K}'}(|X|_H|^2) = \mathbb{M}_H^{\mathcal{K}'}(|X_c|_H|^2) + \|X_d|_H\|^2_{B^2(H,\mathcal{K}')} \; a.s.. \tag{32}$$

In a similar way to (32) one can argue to obtain

$$\mathbb{M}_G^{\mathcal{K}}(|X|^2) = \mathbb{M}_G^{\mathcal{K}}(|X_c|^2) + \|X_d\|^2_{B^2(G,\mathcal{K})} \; a.s.. \tag{33}$$

In addition, from Lemma 7, $\mathbb{M}_H^{\mathcal{K}'}(|X_c|_H|^2) = \mathbb{M}_G^{\mathcal{K}}(|X_c|^2)$ and thus, from equations (32) and (33),

$$\mathbb{M}_G^{\mathcal{K}}(|X|^2) = \mathbb{M}_H^{\mathcal{K}'}(|X|_H|^2) \; a.s.$$

if and only if $\|X_d|_H\|^2_{B^2(H,\mathcal{K}')} = \|X_d\|^2_{B^2(G,\mathcal{K})}$ a.s. which, from Lemma 6, is equivalent to the condition $D_X \bigcap D_X + \lambda = \emptyset$ for all $\lambda \in H^\perp \setminus \{0\}$. □

We can also obtain a condition in terms of filtered versions of X. Namely:

Theorem 7 *Let X be a stationary Gaussian random process. Then: there exists a support of $\mu_{X\,d}$, $D_X \in \mathbf{B}(\widehat{G})$, such that $D_X \bigcap D_X + \lambda = \emptyset$ for all $\lambda \in H^\perp \setminus \{0\}$, if and only if*

$$\mathbb{M}_G^{\mathcal{K}}(|\varphi * X|^2) = \mathbb{M}_H^{\mathcal{K}'}(|\varphi * X|_H|^2) \; a.s. \tag{34}$$

for every $\widehat{\varphi} \in A(\widehat{G})$.

Proof. Define the convolution process $Y = \varphi * X = \varphi * X_c + \varphi * X_d$. Then, if $\widehat{\varphi} \in A(\widehat{G}) \subset L^2(\widehat{G}, \mu_X)$, recalling Example 2.3.4 and equation (22), the spectral representation of the discrete component Y_d is given by:

$$Y_d(t) = \sum_{\gamma \in D_X} \langle t, \gamma \rangle \widehat{\varphi}(\gamma) \Phi(\{\gamma\}) \, .$$

Only if part) Let Y be defined as above. Is immediate that the discrete part of the spectral measure of Y, is given by

$$\mu_{Y\,d}(A) = \sum_{\gamma \in D_X} \mathbf{1}_A(\gamma) |\widehat{\varphi}(\gamma)|^2 \mu_X(\{\gamma\}) \, , \tag{35}$$

and then there exists a support of $\mu_{Y\,d}$, $D_Y \subseteq D_X$, so that $D_Y \bigcap D_Y + \lambda = \emptyset$ for all $\lambda \in H^\perp \setminus \{0\}$. But, recalling Theorem 6, this is equivalent to $\mathbb{M}_G^{\mathcal{K}}(|Y|^2) = \mathbb{M}_H^{\mathcal{K}'}(|Y|_H|^2)$ *a.s.*.
If part) For Y defined as above, we shall construct an appropriate $\widehat{\varphi} \in A(\widehat{G})$ such that $D_X = D_Y$. To simplify, let us enumerate $D_X = \{\gamma_k, \; k \in \mathbb{N}\}$. In addition we can find a family of open neighbourhoods of each $\gamma \in D_X$, say $\{V_\gamma, \; \gamma \in D_X\}$ and such that $0 < m_{\widehat{G}}(V_\gamma) < \infty$. Recalling Proposition 1, for each $k \in \mathbb{N}$, we can find a non-negative $\hat{f}_k \in A(\widehat{G})$ so that:

$$\widehat{f}_k(\gamma_k) > 0 \quad \text{and} \quad 1 \geq \widehat{f}_k(\gamma) \geq 0 \text{ for all } \gamma \in \widehat{G} \, . \tag{36}$$

Now, we can define $\widehat{\varphi} \in A(\widehat{G})$ as

$$\widehat{\varphi}(\gamma) = \sum_{k\in\mathbb{N}} 2^{-k} \hat{f}_k(\gamma)\,. \tag{37}$$

By construction $\widehat{\varphi}(\gamma_k) > 0$, for every $k \in \mathbb{N}$.

Thus, from equation (35), $\mu_{Y\,d}(\gamma_k) > 0$ for $k = 1, \ldots$ or equivalently $D_X = D_Y$. But recalling from the hypothesis that $\mathbb{M}_G^{\mathcal{K}}(|Y|^2) = \mathbb{M}_H^{\mathcal{K}'}(|Y|_H|^2)$ *a.s.* we get, by Theorem 6, that $D_Y \bigcap D_Y + \lambda = \emptyset$ for all $\lambda \in H^\perp \setminus \{0\}$ with $D_Y = D_X$, proving the claim. $\square$

3.1.1 Stability Conditions.

Suppose that X is a Gaussian stationary random process for which some of the previous results hold. We now aim to describe a condition under which the results are still valid for a corrupted or distorted version of X. To illustrate, consider the following simple examples:

Example 1.

Let $G = \mathbb{R}$ (so that $\widehat{G} = \mathbb{R}$), $H = \mathbb{Z}$ and $H^\perp = 2\pi\mathbb{Z}$. In addition, let $f \in L^2(\mathbb{R})$ and let X be a Gaussian stationary random process with *discrete spectrum*. Suppose that we receive a signal Y which is X plus an additive noise N. A reasonable model for N could be the following. If W is a Wiener random measure over $\mathbb{R}$ and independent of X, define $N(t) = \int_{\mathbb{R}} \widehat{f}(\gamma) e^{i\,t\gamma} dW(\gamma)$ and $Y(t) = X(t) + N(t)$ for each $t \in \mathbb{R}$. An easy calculation shows that the spectral measure μ_Y is given by:

$$\mu_Y(A) = \mu_X(A) + \mu_N(A) = \mu_X(A) + \int_A |\widehat{f}(\gamma)|^2 d\gamma\,, \tag{38}$$

for each measurable subset $A \subseteq \mathbb{R}$, and thus $\|\mu_X - \mu_Y\|_{\mathcal{M}(\mathbb{R})} = \int_{\mathbb{R}} |\widehat{f}(\gamma)|^2 d\gamma$. So that Y can be regarded as the original signal corrupted by certain noise N with variance $\|f\|^2_{L^2(\mathbb{R})}$ and, by (38), this value gives an idea of how close is X from Y. In fact, this also equals for all t, the mean square error: $\mathbf{E}|X(t) - Y(t)|^2 = \|f\|^2_{L^2(\mathbb{R})} > 0$, if $f \neq 0$. Suppose that it is also known that

$$\lim_{T\longrightarrow\infty} \frac{1}{2T} \int_{-T}^{T} |Y(t)|^2 dt = \lim_{N\longrightarrow\infty} \frac{1}{2N} \sum_{n=-N}^{N} |Y(n)|^2 \;\; a.s.$$

In this case, under the knowledge that X has originally discrete spectrum we can infer, by Theorem 6, that there exists a support D_X of μ_X verifying that $D_X \bigcap (D_X + \lambda) = \emptyset$ for all $\lambda \in 2\pi\mathbb{Z} \setminus \{0\}$, which is equivalent to $\mathcal{H}(X) = \mathcal{H}(X|_{\mathbb{Z}})$ by Theorem 2.

Example 2.

Let $\gamma_0 \in \widehat{G}$, $\lambda_0 \in H^\perp$ be such that $\mu_X(\{\gamma_0, \gamma_0 + \lambda_0\}) = 0$, and take $\varepsilon > 0$. Another case of perturbation of a random process can be given by the following model. We can consider a random process X with spectral measure Φ_X and and a modified version of it X_ε, with spectral random measure given by:

$$\Phi_\varepsilon(A) = \Phi_X(A) + \frac{\sqrt{\varepsilon}}{2}(\mathbf{1}_A(\gamma_0)C(\gamma_0) + \mathbf{1}_A(\gamma_0 + \lambda_0)C(\gamma_0 + \lambda_0))\,,$$

for any $A \in \mathbf{B}(\widehat{G})$. Where $C(\gamma_0)$ and $C(\gamma_0 + \lambda_0) \sim \mathcal{N}(0, 1)$ i.i.d. complex random variables, and also independent of X. In this case, since $\mu_\varepsilon = \mu_X + \frac{\varepsilon}{4}(\delta_{\gamma_0} + \delta_{\gamma_0+\lambda_0})$, we have that $\|\mu_X - \mu_\varepsilon\|_{\mathcal{M}(\widehat{G})} = \frac{\varepsilon}{2}$.

These examples motivate the following stability conditions. For $A \in \mathbf{B}(\widehat{G})$, define the subsets of $\mathcal{M}(\widehat{G})$ by:

$$\mathscr{K}_d(A) = \{\mu \in \mathcal{M}(\widehat{G}) : \ \ \mu_d(A^c) = 0\} . \tag{39}$$

$$\mathscr{K}(A) = \{\mu \in \mathcal{M}(\widehat{G}) : \ \ \mu(A^c) = 0\} . \tag{40}$$

Recall that $\mathcal{U} \subset \mathscr{K}(A)$ is relatively open, with respect to $\mathscr{K}(A)$, if there exists an open subset $\mathcal{U}' \subset \mathcal{M}(\widehat{G})$ such that $\mathcal{U} = \mathcal{U}' \bigcap \mathscr{K}(A)$. Analogously , we can consider relatively open subsets with respect to $\mathscr{K}_d(A)$. Now, we can prove the following improvement of Corollary 1:

Theorem 8 *Let $A \in \mathbf{B}(\widehat{G})$. Then the following assertions are equivalent:*
(i) $A \bigcap A + \lambda = \emptyset$, for all $\lambda \in H^\perp \setminus \{0\}$.
(ii) There exists a relatively open and non empty subset $\mathcal{U} \subseteq \mathscr{K}_d(A)$ so that

$$\mathbb{M}_G^{\mathcal{K}}(|X|^2) = \mathbb{M}_H^{\mathcal{K}'}(|X|_H|^2) \ a.s.$$

for any Gaussian stationary process X such that $\mu_X \in \mathcal{U}$.
(iii) There exists a relatively open and non empty subset $\mathcal{U} \subseteq \mathscr{K}(A)$ so that

$$\mathcal{H}(X) = \mathcal{H}(X|_H)$$

for any Gaussian stationary process X such that $\mu_X \in \mathcal{U}$.

Proof. (i) $\Longrightarrow$ (ii) For a fix $\rho > 0$, given $a \in A$, consider δ_a and the open ball $B(\delta_a, \rho) \subset \mathcal{M}(\widehat{G})$. Define

$$\mathcal{U} = \bigcup_{a \in A} B(\delta_a, \rho) \cap \mathscr{K}_d(A) . \tag{41}$$

By construction, this subset $\mathcal{U}$ is relatively open. And the claim is an immediate consequence of Theorem 6.

Similarly, (i) $\Longrightarrow$ (iii) is a consequence of Theorem 2 by replacing $\mathscr{K}_d(A)$ by $\mathscr{K}(A)$ in equation (41).

(ii)$\Longrightarrow$(i) If $\mathcal{U} \neq \emptyset$ then there exists a Gaussian stationary random process, say X, such that its spectral measure $\mu_X \in \mathcal{U}$, and $\mu_{X\,d}(A^c) = 0$. Suppose that $A \bigcap (A + \lambda_0) \neq \emptyset$ for some $\lambda_0 \in H^\perp \setminus \{0\}$. Let D_X be any support of $\mu_{X\,d}$ such that $D_X \subseteq A$, then we have two possibilities: $D_X \bigcap (D_X + \lambda_0) \neq \emptyset$ or $D_X \bigcap (D_X + \lambda_0) = \emptyset$. In the first case, by Theorem 6, this is equivalent to $\mathbb{M}_G^{\mathcal{K}}(|X|^2) \neq \mathbb{M}_H^{\mathcal{K}'}(|X|_H|^2)$ with a positive probability. And the result, for this case, is proved.
In the other case, $D_X \bigcap (D_X + \lambda_0) = \emptyset$ for all $\lambda \in H^\perp \setminus \{0\}$, there exists $\gamma_0 \in A \setminus D_X \subset \widehat{G}$ such that $\{\gamma_0, \gamma_0 + \lambda_0\} \subset A \setminus D_X$. For $\varepsilon > 0$ we can define a new stationary Gaussian random process $X_\varepsilon = \{X_\varepsilon(t), t \in G\}$ given by

$$X_\varepsilon(t) = X(t) + \frac{\sqrt{\varepsilon}}{2} \left(C(\gamma) \langle t, \gamma_0 \rangle + C(\gamma_0 + \lambda_0) \langle t, \gamma_0 + \lambda_0 \rangle \right) , \tag{42}$$

for all $t \in G$, with $C(\gamma_0)$ and $C(\gamma_0 + \lambda_0) \sim \mathcal{N}(0,1)$ i.i.d. complex random variables, and also independent of X. (In fact, X_ε has an spectral representation as in Example 2)

Moreover, by a direct calculation, its covariance is

$$R_{X_\varepsilon}(t-s) = \mathbf{E}(X_\varepsilon(t)\overline{X_\varepsilon(s)}) = R_X(t-s) + \frac{\varepsilon}{4} \left(\langle t, \gamma_0 \rangle + \langle t, \gamma_0 + \lambda_0 \rangle \right) .$$

Thus, since the spectral measure of X_ε verifies that $\overset{\vee}{\mu}_\varepsilon = R_{X_\varepsilon}$, then:

$$\mu_\varepsilon = \mu_{X\,c} + \mu_{X\,d} + \frac{\varepsilon}{4} \left(\delta_{\gamma_0} + \delta_{\gamma_0 + \lambda_0} \right) .$$

By construction, a support of the discrete part of μ_ε , D_ε is given by:

$$D_\varepsilon = \{\gamma_0, \gamma_0 + \lambda_0\} \bigcup D_X \subseteq A\,,$$

and consequently,

$$D_\varepsilon \bigcap (D_\varepsilon + \lambda_0) \neq \emptyset\,. \tag{43}$$

Finally, since $\mathcal{U}$ is open, we can fix $\varepsilon > 0$ in (42) such that the open ball $B(\mu_X, \varepsilon) \subseteq \mathcal{U}$. Moreover, $\mu_\varepsilon \in B(\mu_X, \varepsilon)$ since the supports of δ_{γ_0} and $\delta_{\gamma_0+\lambda_0}$ are disjoint, and therefore

$$\|\mu_X - \mu_\varepsilon\|_{\mathcal{M}(\widehat{G})} = \frac{\varepsilon}{4}\,\|\delta_{\gamma_0} + \delta_{\gamma_0+\lambda_0}\|_{\mathcal{M}(\widehat{G})} = \frac{\varepsilon}{4}\left(\|\delta_{\gamma_0}\|_{\mathcal{M}(\widehat{G})} + \|\delta_{\gamma_0+\lambda_0}\|_{\mathcal{M}(\widehat{G})}\right) = \frac{\varepsilon}{2}\,.$$

So, under the assumption that $A \bigcap (A + \lambda_0) \neq \emptyset$, we found a stationary random Gaussian process X_ε such that its spectral measure $\mu_\varepsilon \in \mathcal{U}$ and, recalling Theorem 6, by equation (43) it also verifies: $\mathbb{M}_G^{\mathcal{K}}(|X|^2) \neq \mathbb{M}_H^{\mathcal{K}'}(|X|_H|^2)$ with a positive probability.

(iii)$\Longrightarrow$(i) Is very similar to the previous argument and we will sketch the main steps of the proof. Again, if $A \bigcap A + \lambda_0 \neq \emptyset$ for some $\lambda_0 \in H^\perp \setminus \{0\}$, since $\mathcal{U}$ is non empty there exists a Gaussian stationary random process X such that $\mu_X \in \mathcal{U}$ and therefore it has a support $S_X \subseteq A$. If $\mu_X(S_X \cap S_X + \lambda_0) > 0$ the result is immediate by Theorem 2. Otherwise, we have that $\mu_X(S_X \cap S_X + \lambda) = 0$ for all $\lambda \neq 0$. However, as in the previous proof, we can take the same process X_ε of (42) sufficiently close to X (in the sense of the norm of $\mathcal{M}(\widehat{G})$) such that $\mu_\varepsilon \in \mathcal{U}$. Its spectral measure is given by $\mu_\varepsilon = \mu_X + \frac{\varepsilon}{4}\left(\delta_{\gamma_0} + \delta_{\gamma_0+\lambda_0}\right)$ and thus S_ε a support of X_ε verifies $\{\lambda_0, \lambda_0 + \gamma_0\} \subseteq S_\varepsilon$. Consequently:

$$\mu_\varepsilon(S_\varepsilon \bigcap S_{\varepsilon+\lambda_0}) > \frac{\varepsilon}{4}\delta_{\gamma_0+\lambda_0}(\{\gamma_0 + \lambda_0\}) = \frac{\varepsilon}{4} > 0,$$

and therefore, by Theorem 2, $\mathcal{H}(X_\varepsilon) \neq \mathcal{H}(X_\varepsilon|_H)$. □

An immediate consequence is that if we restrict to the class of Gaussian random processes with discrete spectrum, the three conditions become equivalent.

Corollary 2 *Let $A \in \mathbf{B}(\widehat{G})$. Then the following assertions are equivalent:*
(i) $A \bigcap A + \lambda = \emptyset$, for all $\lambda \in H^\perp \setminus \{0\}$.
(ii) There exists $\mathcal{U} \subseteq \mathcal{M}_d(\widehat{G})$ relatively open with respect to $\{\mu \in \mathcal{M}_d(\widehat{G}) : \ \mu(A^c) = 0\}$ so that:

$$\|X\|^2_{B^2(G,\mathcal{K})} = \|X|_H\|^2_{B^2(H,\mathcal{K}')} \ a.s.$$

for any Gaussian stationary process X such that $\mu_X \in \mathcal{U}$.
(iii) There exists $\mathcal{U} \subseteq \mathcal{M}_d(\widehat{G})$ relatively open with respect to $\{\mu \in \mathcal{M}_d(\widehat{G}) : \ \mu(A^c) = 0\}$ so that:

$$\mathcal{H}(X) = \mathcal{H}(X|_H)$$

for any Gaussian stationary process X such that $\mu_X \in \mathcal{U}$.

4 Appendix

4.1 The proof of Lemma 1.

Lemma 3, p.236 of the classic book [11], contains a proof of this result for a Real interval. However, in the author's opinion, the final steps of this proof contains a small gap since it involves the integration of a not necessarily measurable function. In order to make the presentation self contained, we present the following adaptation of the argument presented there.

Proof. Let

$$\varphi(t,\gamma) = \sum_i \alpha_i \mathbf{1}_{R_i}(t,\gamma) \tag{44}$$

be a simple function, with $R_i = F_i \times B_i \in \mathbf{B}(G) \otimes \mathbf{B}(\widehat{G})$ a measurable rectangle. Then, by the definition of the stochastic integral, if $\omega \in \Omega$: $\int\limits_{\widehat{G}} \varphi(t,\gamma) d\Phi_X(\gamma)(\omega) = \sum\limits_i \alpha_i \mathbf{1}_{F_i}(t) \Phi_X(B_i)(\omega)$, which is clearly a $\mathbf{B}(G) \otimes \mathcal{F}$-measurable. Now, let $\varphi \in L^2(G \times \widehat{G}, \mathbf{B}(G) \otimes \mathcal{F}, \nu \otimes \mu)$, there exists a sequence of $\{\varphi_n\}_{n\in\mathbb{N}}$ of simple functions, of the form as (44), such that $\lim_{n\longrightarrow\infty} \|\varphi_n - \varphi\|_{L^2(G\times\widehat{G},\mathbf{B}(G)\otimes\mathcal{F},\nu\otimes\mu)} = 0$. Therefore $\{\varphi_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Noting that, for each n, is defined $\int_{\widehat{G}} \varphi_n(t,\gamma) d\Phi_X(\gamma)$ and is a measurable function:

$$\lim_{n\longrightarrow\infty} \|\varphi_n - \varphi_m\|_{L^2(G\times\widehat{G},\nu\otimes\mu)} = \int\limits_G \mathbf{E} \left| \int_{\widehat{G}} (\varphi_n(t,\gamma) - \varphi_m(t,\gamma)) d\Phi_X(\gamma) \right|^2 d\nu(t)\,.$$

Then there exists Y a $\mathbf{B}(G) \otimes \mathcal{F}$-measurable function such that

$$\lim_{n\longrightarrow\infty} \int\limits_G \mathbf{E} \left| \int_{\widehat{G}} \varphi_n(t,\gamma) d\Phi_X(\gamma) - Y(t) \right|^2 d\nu(t)\,.$$

On the other hand, for each $t \in G$: $\lim\limits_{n\longrightarrow\infty} \mathbf{E} \left| \int_{\widehat{G}} \varphi_n(t,\gamma) d\Phi_X(\gamma) - X(t) \right|^2$, where $X(t) = \int_{\widehat{G}} \varphi(t,\gamma) d\Phi_X(\gamma)$. Note that for m_G-a.e. in $t \in G$: $\mathbf{P}(X(t) = Y(t)) = 1$, since ν and m_G are equivalent measures. Now, for each $(t,\omega) \in G \times \Omega$ define a $\widetilde{\sigma}(\mathbf{B}(G) \otimes \mathcal{F})$-*measurable* process $\widetilde{X}$ in the following way:

$$\widetilde{X}(t,\omega) = Y(t,\omega) \mathbf{1}_{\{t:\mathbf{P}(X(t)=Y(t))=1\}} + X(t,\omega) \mathbf{1}_{\{t:\mathbf{P}(X(t)=Y(t))<1\}}$$

This measurable process verifies that $\mathbf{P}(X(t) = Y(t)) = 1$ for all $t \in \mathbb{G}$. □

4.2 More on Complex Stationary Random Processes.

Recall from Section 2.3.2, that a complex Gaussian stationary random process, that X is decomposed in $X(t) = X_1(t) + iX_2(t)$, where X_1 and X_2 are two real stationary (cross)correlated stationary random processes. Recall that in this case there exists two cross-spectral measures $\mu_{i\,j}$ such that the cross correlations verify $\mathbf{E}(X_i(t)X_j(s)) = \int\limits_{\widehat{G}} \langle t-s, \gamma \rangle d\mu_{i\,j}(\gamma)$. Usually, in the applied literature it is claimed that condition (13) is preserved by linear operations. In this direction, we state the following straightforward result adapted for our context and for which no reference was found. Its proof is left for the reader.

Lemma 10 *Let X be a complex w.s.s. stationary random process. The following statements are equivalent:*
(i) X verifies condition (13).
(ii) $\mu_{1\,1} = \mu_{2\,2}$ *and* $\mu_{1\,2} = -\mu_{2\,1}$.
(iii) $\mathbf{E}(Z_1 Z_2) = 0$, *for every* $Z_1, Z_2 \in \mathcal{H}(X)$.

In the Gaussian case, note that (iii) implies that if in addition $\mathbf{E}(Z_1\overline{Z_2}) = 0$ then the random variables Z_i are independent. For example if $f_i \in L^2(\widehat{G}, \mathbf{B}(\widehat{G}), d\mu_X)$, $i = 1,2$ are such that $\{f_1 \neq 0\} \cap \{f_2 \neq 0\} = \emptyset$ then the random variables $Z_i = \int\limits_{\widehat{G}} f_i d\Phi_X$, $i = 1,2$ are independent. This fact was extensively used throughout the present article.

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