Fiction, possibility and impossibility: Three kinds of mathematical fictions in Leibniz's work

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Abstract

This paper is concerned with the status of mathematical fictions in Leibniz's work and especially with infinitary quantities as fictions. Thus, it is maintained that mathematical fictions constitute a kind of symbolic notion that implies various degrees of impossibility. With this framework, different kinds of notions of possibility and impossibility are proposed, reviewing the usual interpretation of both modal concepts, which appeals to the consistency property. Thus, three concepts of the possibility/impossibility pair are distinguished; they give rise, in turn, to three concepts of mathematical fictions. Moreover, such a distinction is the base for the claim that infinitesimal quantities, as mathematical fictions, do not imply an absolute impossibility, resulting from self-contradiction, but a relative impossibility, founded on irrepresentability and on the fact that it does not conform to architectonic principles. In conclusion, this "soft" impossibility of infinitesimals yields them, in Leibniz view, a presumptive or "conjectural" status.

Keywords
Fictions / Infinitesimals / Infinitary quantities / Possibility / Impossibility

1. Introduction

In this paper we will deal with the question of the status of the concept of mathematical fiction in Leibniz's work, with special emphasis on the application of this concept to infinitary quantities. Within this framework, our analysis holds that mathematical fictions constitute a kind of symbolic notion that implies various degrees of impossibility. From this perspective, we propose to examine different kinds of notions of possibility and impossibility, introducing clarifications in relation to the usual interpretation of both modal concepts, which is usually based on the property of non-contradiction or consistency. Thus, we distinguish three concepts of the possibility/impossibility pair, which give rise, in turn, to three concepts of mathematical fictions. The distinction between different classes of mathematical fictions provides us with the basis to hold that infinitesimal quantities, as mathematical fictions, do not imply an absolute impossibility, resulting from self-contradiction, but a relative impossibility, founded on irrepresentability and the fact that it does not conform to architectonic principles. As we will see, this "soft" concept of the impossibility of infinitesimals as fictions will mean that Leibniz assigns them a presumptive or "conjectural" status.

In order to approach the analysis of mathematical fictions in terms of symbolic notions or concepts, we will also synthetically deal with some of the main concepts of Leibniz's conceptions about symbolic knowledge. Briefly, the Leibnizian concepts concerning the connection between signs and knowledge lead to a methodological approach to mathematical fiction. Thus, a mathematical fiction turns out to be a class of symbolic notion or concept[1] with the following features or fundamental notes (Raffo Quintana 2020, pp. 131-150):

1. In principle it is an empty concept, without a denotation or an idea that corresponds to it. In the Leibnizian classification of notions, it would be considered a confused notion (Esquisabel 2012a, pp. 4-7).
2. It is analogical in nature, in the sense that its introduction is based on an analogical relation to concepts of entities and operations already known or established (Sherry & Katz 2012, pp. 166-192; Esquisabel 2020).
3. As with any symbolic notion, a fiction functions as a surrogate, in the sense that it is used instead of something else. Unlike the symbolic notions that substitute the consideration of the object as such (or of the idea of the object), fictions surrogate procedures or operations. In relation to this surrogative function, the relation that a fiction maintains with what is surrogated is, in our opinion, twofold. On the one hand, a fiction provides an abbreviation of the mathematical procedure and, in that sense, it works as a compendium. On the other hand, the surrogated procedure has a foundational nature, in the sense that it is exact and rigorous (it does not appeal to fictions). Thus, the result obtained through fiction should in principle always be able to be validated by means of the corresponding procedure (for the symbolic notion in general and its function for substitution and abbreviation, see Esquisabel 2012a and for the case of infinitesimals, see Raffo Quintana 2020).
4. Its introduction is in general informal in nature, but it can be made more rigorous by means of syntactic procedures regulated by operating rules. In this way, Leibniz tries to construct a calculus that regulates the operation with fictitious notions.
5. A fiction has a heuristic power, in the sense that it broadens the scope of the "art of invention": it provides better solutions to known problems, in that they are simpler and more elegant ones. It also broadens the domain of soluble problems, since it provides a solution to problems for which a solution had not yet been found (Sherry & Katz 2012, pp. 181-190).

The properties that we have just stated concerning the methodological efficacy of mathematical fictions constitute the general framework that guides our investigation; however, it is beyond the scope of this paper to systematically consider them in their

[1] An analysis of the symbolic notion or concept can be found in Esquisabel (2012a, pp. 1-49). This topic connected to the question of the fictionalism of infinitesimals was dealt with in Esquisabel (2012b), and more recently it can be found in Rabouin and Arthur (2020, pp. 406-407).

entirety. Accordingly, we will limit our considerations mainly to the first statement, namely, that mathematical fictions are symbolic notions, although in the development of our analysis we will refer to the remaining features.

Thus, we hold that Leibniz introduces infinitesimal quantities on the basis of these properties as regulatory principles. For the moment, we will elucidate the concept of the infinitesimal and other related concepts in terms of their status as mathematical fictions, providing some simple examples of their methodological function. Additionally, our approach, which adopts a semiotic perspective, will only tangentially affect the well-known controversy regarding the nature of infinitesimals, as well as the justification for their introduction,[2] whether the defended approach –just to name the main lines of interpretation– is a "syncategorematic" one (Ishiguro 1990, chap. V; Arthur 2013, pp. 553-593; see also Arthur 2018, pp. 155-179), an "ideal" one (Sherry and Katz 2012, Bair et al. 2018, pp. 186-224, contains a good summary of the current discussion on the question of the status of Leibnizian infinitesimals), an "epsilontic" one (Knobloch 1994, pp. 265-278; Knobloch 2002, pp. 43-57) or an approach based on the principle of continuity (Bos 1974, pp. 1-90). On the contrary, we are more interested in determining how Leibniz himself conceived and tried to interpret the novelty of his method.

2. Symbolic knowledge, symbolic cognition and fictions

As anticipated, we will deal with Leibniz's understanding of mathematical fictions, by departing from his conceptions about symbolic knowledge and taking as a starting point the examination he developed in *Meditationes de cognitione, veritatis et ideis* (MCVI), from 1684 (A VI 4, 585-492[3]). It is well-known that in this brief but central work Leibniz paradigmatically presents his classification of knowledge or notions, among which can be found the confused notion, which is what interests us in the present context. Thus, Leibniz firstly distinguishes between clear and obscure notions; in turn, within clear notions, he establishes a division between confused and distinct notions, while the latter are divided into adequate and inadequate ones. This last division is finally connected with the distinction between intuitive and symbolic notions (for a review of this way of understanding the division, see Esquisabel 2012a, pp. 5-7). Thus, a symbolic notion is characterized by being a sign of a sensible nature that substitutes, in one way or another, the comprehension of a notion or idea.[4] Although Leibniz seems to oppose symbolic notions to intuitive notions only, it is wrong to interpret symbolic notions as merely replacing the use of intuitive notions. As we will see, the use of purely symbolic notions implies the epistemic risk of accepting notions containing a contradiction since they are "cognitively confused", that is, because we have not analyzed them properly, as occurs with the notion of maximum speed.

Although the distinction is not always established, it is convenient to differentiate between symbolic knowledge itself, on the one hand, and symbolic cognition, on the other. By the former, we understand the true information that we can obtain through the use and

[2] For a summary of the controversy in the seventeenth century, see Mancosu (1996, chap. 6) and Jesseph (1998, pp. 6-38). For the controversy with Newton regarding the attribution of its originality, see Sonar (2016).

[3] We will refer to Leibniz (1923-seq.) following the standard abbreviation: A, followed by series (in Roman numerals), volume (in Arabic numerals) and page number. Ex.: A VII 6, 600.

[4] For the moment, we consider both concepts as equivalent, but later it will be necessary to distinguish them.

operation with different forms of semiotic representation, while symbolic cognition consists of the cognitive operations, such as to reason, infer, remember, imagine, etc., that we carry out or can carry out with the assistance of signs. This distinction is important to differentiate between the algorithmic manipulations of signs and the cognitive orientation that the use of semiotic resources can provide.

The symbolic notion, which has an ambivalent status in relation to the previous distinction, is defined as a representation or thought with a perceptible material support, that is, it has semiotic features. The fundamental feature of a symbolic notion, as a sensible sign, is given by its various functions as a support for cognition. On the one hand, it accompanies and supports the comprehension of simple elements of thought. On the other, when the comprehension is composed of a multiplicity of contents of thought, the symbolic concept substitutes or surrogates the consideration of each thought or globally "embraces" the conceptual components taken together.[5] Accordingly, the symbolic notion is known as a "blind notion" or "blind thought". It is worth clarifying that Leibniz frequently refers to "blind notion" or "blind concept" and not to "symbolic" notion or concept (as in *MCVI*), including a wide range of notions within blind notion or concept, as, for example, notions we apply in our common languages, most of which are confused notions (or "distinct-inadequate" ones, in the sense of *MCVI*) (see A VI 1, 170, 551; A VI 2, 481; A VI 4, 587; A VI 6, 185-186, 259, 275, 286, *inter alia*). In this way, the confused notion is essentially "blind" or "symbolic", in the paradoxical sense that it is a cognition "without concepts", that is, without capturing "intellectual contents" or "ideas".

Additionally, a symbolic notion can be completely separated from the acts of cognition or of understanding of meaning, in such a way that it can be treated as a purely physical object, according to syntactic or combinatorial rules. In that case, we have a "blind notion" in the proper sense of the expression, since in this way the symbolic notion can be operationally manipulated, as usually occurs in the calculus of algebra and of arithmetic, which allows surrogative inferences to be made. On the other hand, Leibniz bases the surrogative ability of semiotic forms on the possibility of establishing structural analogies between such forms and what is represented (Esquisabel 2012a, pp. 18-32 and 32-43. Cf. Swoyer 1991 and 1995). Thus, the introduction of mathematical fictions constitutes a challenge for this way of conceiving the efficacy of symbolic notions, since their efficiency and soundness for surrogative inferences must be justified. As we have anticipated, the analysis of this question, which corresponds to features 3 to 5 on our list, is beyond the scope of this paper.

Using this framework, we propose to elucidate the notion of fiction in terms of a symbolic notion without denotation. To do this, we will appeal to the distinction introduced by Hans Poser between idea, on the one hand, and notion or concept, on the other (Poser 1979, pp. 309-324, Poser 2016, pp. 90-92). This distinction was indeed introduced by Leibniz in the 1680s. For example, in *MCVI* he clearly states the possibility of a thought without ideas (A VI 4, 588). Similarly, in *Quid sit idea*, a text written a few years before *MCVI*, Leibniz maintains a conception of ideas in terms of "faculties" or dispositions to think about something (A VI 4, 1370). The idea as a faculty reappears in the *Discourse on metaphysics*, where Leibniz contrasts idea with notion, conceiving the former as an active

[5] This phenomenon of vague or "global" understanding occurs mainly in verbal language, although it does not necessarily have to accompany every use of symbolic notions. See Esquisabel (2012a, pp. 10-18), where the distinction between two kinds of symbolic thought is proposed.

power of thinking about an object, while the latter is described as what is formed in thought as a result of an actualization of the former (A VI 4, 1572; see also A VI 4, 591; A VI 6, 12; A VI 6, 109). Thus, every idea is *expressed* –in the Leibnizian sense of the term– in notions. On the other hand, although for every idea there may be a multitude of notions that express it, the reverse does not apply; in other words, there is not always an idea for every actual notion. There may be notions without ideas and in that case they are "false" notions, as, for example, when the notion we think about implies a contradiction, according to the classic example invoked by Leibniz of maximum speed. In this way, we maintain the thesis that an idea fulfills the role of being the reference or denotation of a notion. In other words, a notion refers to or denotes an object –when there is one– only by means of an idea, which is its immediate denotation. This is how we understand the Leibnizian statement that an idea is an immediate internal object (A VI 6, 109). Thus, a "false" symbolic notion, that is, one without ideas, lacks reference or denotation. In conclusion, fictions and especially mathematical fictions can be considered as a class of "empty" symbolic notions, that is, devoid of reference.

3. Fiction as a symbolic notion

According to the results of the previous sections, we will characterize a fiction *in principle* as a blind or symbolic notion, cognitively confused, in such a way that, when we distinctly analyze it, we notice that it is empty or without denotation, either because the corresponding idea is impossible ("false" or "non-existent"), or because the object that corresponds to the notion has not existed, does not exist nor will it actually exist. Therefore it is necessary to distinguish two kinds of fictions: the "ideal" one, which violates the principle of contradiction (for example, the notion of a round square), and the "factual" one, which refers to objects that do not exist in reality, as is the case of fictional characters in novels. For the moment, this distinction will suffice, although we must introduce some clarifications on the concept of "ideal" fiction, as we will see later, after carrying out a subtler analysis of the concept of impossibility (see also Rabouin and Arthur 2020, p. 407). It is also worth mentioning that this distinction does not affect mathematical objects with a consistent definition (such as circle), since they "exist" as *possibilia* in the mind of God, even if they do not actually exist. On the contrary, existence does not correspond to fiction in any sense.

In a short fragment dedicated to the Stoic argument of *sorites* entitled *Acervus Chrisyippi* (1678, A VI 4, 69-70), Leibniz briefly thinks about what we would today call "vague" concepts or terms, that is, those having cases in which the application of the concept or term is indeterminate, such as poverty, baldness, heat, cold, warmth, etc. Leibniz considers that all these notions are imaginary in nature, and are characterized by him in this way:

> [All these notions] taken absolutely, are vague imaginary notions, indeed false ones, that is, ones having no corresponding idea. (…) I call those notions imaginary which are not in the things outside us, but whose essence is to appear to us to be something. (A VI 4, 69-70. Translation: Leibniz (2001, p. 231), slightly modified)

It is true that Leibniz is not talking here about fictions, but about imaginary notions. Although a close connection between the fictionality and the (purely) imaginary could be shown, for the sake of brevity allow us to argue that fictions fall within the field of imagination. In any case, the lack of ideas is precisely what we want to highlight in the present context. It is clear that to have a notion is not the same as to have an idea and in that sense a notion can be false. This is precisely the main feature of a fiction.

Now, if a fiction is a notion devoid of any idea and therefore a false one, we can nevertheless detect two possible cases of falsehood or of "lack of denotation". The first corresponds to the strictly speaking impossibility (based in principle on inconsistency or "contradictoriness", although we will see later that there are other kinds of impossibility regarding the existence of mathematical objects), and the second one corresponds to the factual non-existence. We could say that it is about the ontological limitations of fictions. In *De ente, existent, aliquo, nihilo et similibus*, a list of definitions dated between 1683 and 1685/86, we find precisely this characterization of a fiction:

> A *fiction* is the thought of an impossible thing, such as the fastest motion; sometimes it is also taken as the concept of a thing that never existed, as *Argenis*. (A VI 4, 570)

As we will see later, the distinction made by Leibniz between two kinds of fictions is essential for distinguishing the kinds of mathematical fictions. In this case, the "factual" fictions consist of notions of objects that never existed, as in the case of Barclay's *Argenis*. However, this class of fictions does not exclude the possibility of existing in the future, since they do not imply in themselves a contradiction, as is the case of the former fictions. For the case of some kinds of mathematical fictions, such as infinitesimal quantities, a stricter condition is required: although they are not in themselves inconsistent, they cannot exist in any way, at least in the actual world as it is constituted. We will discuss this later.

Finally, a fiction implies a certain confusion or lack of distinction. As we pointed out before, the confused notion or concept is closely connected to that of blind or symbolic thought, especially when considering signs of a verbal nature, such as those that generally constitute phonic languages. Our cognitive limitations, whether in regard to intellection, memory, or imagination, set limitations on operations with compound notions, as are most of the concepts that we apply in our cognitions. Without the intention of being exhaustive here, it is possible to detect in Leibniz's conception of the confused notion different degrees of confusion, so to speak. Thus, depending on whether the confusion is ultimately solvable or not, "cognitively" confused notions are distinguished from "essentially" confused notions. Thus, for example, the concept of cognitively confused cognition is present in the following text of the *New Essays*:

> If I am confronted with a regular polygon, my eyesight and my imagination cannot give me a grasp of the thousand which it involves: I have only a *confused* idea[6] both of the figure and of its number until I *distinguish* the number by counting. (NE, A VI 6, 261; Translation: Leibniz (1996, p. 261)).

This kind of confusion can be solved through an adequate analysis of the component notions. However, what may happen is that in the final analysis the notion is found to be inconsistent or impossible. In that case, its elucidation will reach the proof of its falsity:

> Firstly, there is what is thinkable, which it is impossible, if it involves a contradiction, when it is distinctly thought, even though it could be confusingly thought. (A VI 4, 388. See also A VI 3, 276-277 (for the Parisian period); A VI 4, 590; A VI 4, 199; A VI 4, 1500).

[6] Leibniz is not always terminologically consistent in relation to the distinction between notion and idea. Here "idea" must be understood in the sense of "notion" or "concept".

On the other hand, there are essentially confused notions which cannot be analyzed in their component notions, due to the very limitations of our cognitive capacities (as, for example, of our senses), as occurs with the data of sensation, that is, colors, smells or tastes (*MCVI*, A VI 4, 586).

Our interest focuses on "cognitively" confused notions, since they normally intervene in our language, whether it be oral or written. Thus, the notion merges with the cognitive meaning which usually accompanies a word or sentence, when the notions that intervene in it are not distinctly analyzed. According to Leibniz terminology, we can speak of a "blind" or symbolic notion, merging the sign and its confused meaning into one thing.
In conclusion, a fiction is a confused symbolic notion, devoid of denotation or idea and, therefore, false. According to what we have examined so far, its falsity can be proven either by an analysis that shows the inconsistency of the notion, that is, its impossibility, or in a factual way, showing that the denoted object does not exist. Hence, it seems that there are two kinds of fiction, namely, the inconsistent and the "factual" one. However, our examination regarding mathematical fictions will show that Leibniz more or less consistently recognized a third possibility: fictions that, without being inconsistent, are impossible because they are geometrically irrepresentable or because they violate architectonic principles. In our interpretation, infinitesimal quantities are fictions of this second and third kind rather than of the first. In this point our opinion differs from that of Rabouin and Arthur (2020), since they maintain that Leibniz rejects the existence of infinitesimals based on the contradictory nature of the concept of infinitesimal, and hence it would be impossible in the strict or "strong" sense. They base their claim on applying Leibniz's thesis on the inconsistency of the infinite number to infinitary concepts, which in turn results in the rejection of infinite wholes (2020, pp. 406-407, 413, 434, 441). Regardless of the fact that we agree on many aspects with Rabouin and Arthur's interpretation, we disagree with the reasons they gave for the Leibnizian rejection of the existence of infinitesimals, and in our opinion the texts they refer to in order to support their interpretation are not convincing. Since we argue that Leibniz did not consider the concept of infinitesimal as *self-contradictory*, we try to provide an alternative conception of impossibility. As the authors themselves admit, Leibniz does not explicitly and openly maintain the contradictoriness of the concept of infinitesimal quantities (2020, p. 422). On the other hand, as we try to show in this paper, the reasons for the impossibility of infinitesimals and for other infinitary concepts are not based on inconsistency, but on architectonic principles. It is true that there are some texts in which Leibniz seems to suggest an argument based on inconsistency, as in the case of *Numeri infiniti* (A VI 3, sp. 503-504); however, as we try to show in this and other papers of ours, if we take into account the reasons that Leibniz mostly gave for rejecting the existence of infinitary quantities, such as infinitely small quantities or infinite bounded lines, we must recognize that arguments based on inconsistency are conspicuous by their absence. On the other hand, it is not our intention to *construct* a possible argument against the existence of infinitesimals based on some concept of Leibniz, whether Leibniz formulated it or not, but to trace the reasons that he explicitly formulated for rejecting the existence of infinitary quantities.

4. Leibniz on the fictionality of infinitary concepts

It is usually argued that Leibniz assumes a fictionalist conception of infinitary concepts from the controversy raised by the diffusion of his method after the publication of his *Nova methodus pro maximis et minimis* (1684, see n. 8) (For example, Jesseph 1998, pp. 16 et seq.; 2008, pp. 225-228; 2015, pp. 192-195 and 200-203). This would be the way in which Leibniz responds to Nieuwentijt's objections and to the controversy raised between the defenders of the new method, for example, Varignon, the Bernoulli brothers and the Marquis de l'Hopital, on one hand, and their detractors, led by Rolle, on the other. There are two texts that constitute commonplaces of the thesis of fiction. The first one can be found in a letter to Des Bosses dated March 11, 1706:

> Speaking philosophically, I no more support infinitely small magnitudes than infinitely large ones, or no more infinitesimals than infinituples. For I consider both to be fictions of the mind, due to abbreviated ways of speaking, which are suitable for calculation, in the way that imaginary roots in algebra are. Moreover, I have demonstrated that these expressions have a great usefulness for shortening thinking, and thus for discovery, and that they cannot lead to error, since it would suffice to substitute for the infinitely small as small a magnitude as one wishes, so that the error would be less than any given; whence it follows that there can be no error. (GP 2 305. Translation: Leibniz 2007, p. 33).

A similar but more concise argument can be found in Leibniz's letter to Varignon of April 14, 1702:

> As for the rest, some years ago I had written to Mr. Bernoulli of Groningen that the infinities and the infinitely small could be considered as fictions, similar to imaginary roots, without this being prejudicial to our calculations, being these fictions useful and well-founded in reality of things. (GM 4 98)

These passages, which summarize many of the features we have conferred to fictions in the introductory section, are representative of what seems to be Leibniz's mature position, after the litmus test of making public his method. However, a passage from a letter to Bernoulli of July 29, 1698 shows that the thesis of fiction was in the very beginnings of the infinitesimal method. In this letter Leibniz indeed confesses to Bernoulli:

> But talking among us, I add the following, which I wrote long time ago in that unpublished treatise, namely, that it can be doubted that infinite in length but bounded straight lines actually exist; for the calculus, however, it is enough that we imagine them, the same as with imaginary roots in algebra. (GM II 524)

The "unpublished treatise" to which Leibniz refers is *De quadratura arithmetica circuli, ellipseos et hyperbolae cujus corollarium est trigonometria sine tabulis* (1676, A VII 6 520-676), completely edited and published for the first time by E. Knobloch in 1992 (Leibniz 1992. French translation: Leibniz 2004; German translation: Leibniz 2016; Spanish translation: Leibniz 2014, pp. 107-241). As it was revealed from Knobloch's and other scholar's studies, this work contains a systematic study of conics by introducing infinitesimal methods, though not the formalism of infinitesimal calculus. In that work we can find the following comment about the introduction of infinitary notions:

> It does not matter whether such quantities [namely, infinitely and infinitely small ones] exist in nature or not; it is enough to introduce them as a fiction, insofar as they

> provide abbreviations for expressing, thinking, and finally for both inventing and demonstrating. (A VII 6, 585)

In other words, as Arthur (2009, pp. 11-28) pointed out, more than twenty years before his letter to Johann Bernoulli, Leibniz already held the fictional character of infinitary concepts, which is a clear sign that his birth certificate had a decisive instrumental and pragmatic orientation. Thus, long before the controversy about the reliability of the infinitesimal calculus, Leibniz expressed a pragmatic point of view about infinite and infinitesimal quantities. At the same time, he expressed serious doubts that objects of this sort had any kind of reality, as the following passage of the *Pacidius Philalethi* (written at the end of 1676), which is consistent with the above text, shows:

> 1 would indeed admit these infinitely small spaces and times in geometry, for the sake of invention, even if they are imaginary. But I am not sure whether they can be admitted in nature. (A VI 3, 564-565. Translation: Leibniz (2001, p. 207))

Leibniz was certainly not the first mathematician who employed infinitary notions, whether they are called indivisible or "infinitely small quantities". Cavalieri and Galileo Galilei had already appealed to the notion of "indivisible" for their mathematical proofs, while Pascal, Roberval, Barrow, Wallis, Fermat, and Newton (see Jullien 2015 for updated studies of the different infinitesimal methods employed by the referred authors) had used some version of the infinitely small for dealing with mathematical problems. In summary, the use of indivisibles or infinitely small quantities was widely extended at the time, and the main problem was not regarding its effective application for demonstrations and resolutions of mathematical problems, but the technical question of the best way to introduce and operate with them. To this rather technical question, the ontological problem about whether such infinitesimal objects were really existent or if on the contrary they had a merely instrumental or methodological status, was added.

Leibniz's works on infinite mathematics provides many examples of the way in which he introduces infinitary concepts as methodological resources for dealing with geometric and arithmetic problems. A classic case is the proof that for certain hyperbolas the infinite space limited by the branch of the hyperbola and the asymptote is equivalent to the area of a finite rectangle (for the demonstration, see *DQAC*, A VII 6, 578-580, Knobloch 1993, p. 84 and Knobloch 1994, p. 275). More generally, it is a key resource in Leibnizian methodology for squaring and determining tangents to conceive a curve in terms of an infinite polygon with infinitesimal sides, as well as the introduction of "characteristic" triangles with infinitesimal sides is (*Methodi tangentium inversae exempla* A VII 5, 326, *inter alia*). In turn, the dealing with infinite series offers cases of application of infinitary procedures to arithmetics, such as the dealing with the sum of the reciprocals of triangular numbers or with the quadrature of the circle by means of the series of reciprocals of the sequence of odd numbers (*Accessio ad arithmeticam infinitorum* A II 342-356; A VII 3 365-369 and 712-714; *DQAC* VII 6, 600 and GM 5 121). In both cases, the dealing with arithmetic series depends on conceiving series as a given totality with infinite terms, something that must be rejected from the strictly philosophical point of view (for the problems that Leibniz finds in infinite series as wholes, see *Numeri infiniti*, A VI, 3 502-503; see also Esquisabel and Raffo Quintana 2017, pp. 1319-1342; Raffo Quintana 2018, pp. 65-73 and Crippa 2017, pp. 93-120). Finally, Leibniz also proposes an algebra of infinitely small quantities, which can be subjected to algebraic operations in the same way as common quantities. In this case, as is known, the differential notation dx and dy is introduced in equations to express infinitesimal increments of finite quantities. Differential quantities can have many interpretations, being able to designate, among other things, infinitely small geometric lines in geometric diagrams. It is common knowledge that Leibniz published the rules of infinitesimal calculus for the first time in *Nova methodus pro maximis et minimis* of 1684,[7] although he had already come up with them by around 1680.

In order to illustrate how Leibniz applies infinitary quantities as fictions, we will exhibit two examples, one geometric in nature and the other emerging from the infinitesimal calculus or "algebra of infinitesimals". For the first one, we consider a part of the proof of proposition XXI of *DQAC*, which refers to the dimensions of a rectangle formed by the abscissa and ordinate of a hyperboloid. This rectangle has the distinctive feature that the abscissa is infinitely small, while the ordinate has an infinite length, although it has an extreme or limit (that is, it is a *linea infinita terminata*) (A VII 6, 548-549; Leibniz 2004, 99-101). As Leibniz himself maintains, both are mathematical fictions. Thus, in Proposition XXI Leibniz says:

> The rectangle 0C0GA0B with the infinitely small abscissa A0B times the infinitely large ordinate 0B0C of the hyperboloid 0C1C2C is an infinite quantity when the order of elevation of the abscissas is greater than the order of elevation of the ordinates in relation of proportionality; if, on the contrary, the order of elevation [of the abscissas] is smaller, the rectangle will be an infinitely small quantity. Finally, if both orders are equal, the rectangle will be an ordinary finite quantity. (*DQAC*, A VII 6, 579. Our translation is based on Leibniz 2004, p. 167)

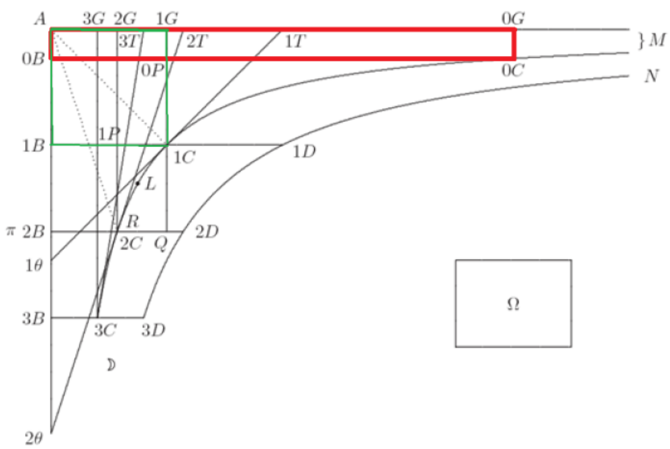

Image: A VI 6, 560

This is a theorem on hyperboloids in which ordinates and abscissas are related as $y^m \cdot x^n = a$, in such a way that $y^m = \frac{a}{x^n}$. Thus, being *AB* the abscissas axis and *AG* the ordinates axis, the rectangle *A0B0C0G* (in red) is constituted by an infinitely small abscissa *A0B* (or *0C0G*) and an infinitely long ordinate *0B0C* (or *A0G*), in which a last term *0B* (or *0G*) at infinity must be assumed. Hence, the theorem states the dimensions of this rectangle, which we will characterize as "infinite-infinitesimal" and we will designate with the letter

[7] Its full title is *Nova methodus pro maximis et minimis, itemque tangentibus, quae nec fractas nec irrationales quantitates moratur, et singulare pro illis calculi genus*, GM 5 220-233, originally published in *Acta Eruditorum*, 1684. French translation: Leibniz (1995, pp. 104-117).

R, in connection with the relation that the exponents of the abscissa and the ordinate maintain to each other. In this way, three possibilities arise (Knobloch 2002, p. 69):
1. if $m < n$, then *R* is infinite
2. if $m > n$, then *R* is infinitely small
3. if $m = n$, then *R* is a finite

For the sake of brevity, we will exhibit Leibniz's proof of the first case, m < n (see Knobloch, 1993, p. 84 and Knobloch, 1994, pp. 273-276). We then have that for every abscissa and ordinate we must prove that the infinite-infinitesimal rectangle *0C0G0A0B* = *A0B0C0G* is of an infinite dimension (A VI 6, 579). Leibniz does not demonstrate the property for every possible exponent, but provides the proof for the particular case of n = 2 and m = 1 and assumes that the procedure is generalizable to any power.
(1) By the general relation between abscissas and ordinates

$$BC = \frac{a}{AB^2} \ or \ BC \,.\, AB^2 = a$$

(2) From *(1)* is obtained that

$$\frac{0B0C}{1B1C} = \frac{A1B^2}{A0B^2}$$

(3) On the other hand, the rectangles *0C0GA0B = A0B0C0G (R)* and *A1B1C1G (in green)* are in the relation

$$\frac{A0B0C0G}{A1B1C1G} = \frac{A0B\,.\,0B0C}{A1B.\,1BIC} = \frac{A0B}{A1B}\,.\,\frac{0B0C}{1B1C}$$

(4) Thus, replacing in *(3)* by *(2)*, we obtain that

$$\frac{A0B0C0G}{A1B1C1G} = \frac{A0B}{A1B}.\frac{A1B^2}{A0B^2} = \frac{A1B}{A0B}$$

Thus, we have an important step for the demonstration, namely, that the infinite-infinitesimal rectangle *R* is to the finite rectangle *A1B1C1G* in the same ratio as the finite abscissa *A1B* with the infinitely small one *A0B*. The sought conclusion is obtained by stating a lemma on the relation between finite, infinite and infinitely small quantities:

(5) Something that has the same ratio to a finite quantity a a finite term to an infinitely small one, or also, as an infinite term to a finite one, is infinite. (A VII 6, 579; Leibniz 2004, p. 169. Our translation is based on Leibniz 2004, p. 167)

Knobloch (1994, p. 273; see also 1993, p. 84 and 2002, pp. 67-68) formulates this lemma as a corollary of an operation rule with finite, infinite, and infinitely small quantities:

finite : infinitely small = infinite : finite = infinite.
Corollary: if finite : infinitely small = x : finite, then x is infinite.

(6) The conclusion obtained in *(4)* fulfills the condition of the lemma *(5)*, since the ratio between the infinite-infinitesimal rectangle *R* and the finite rectangle *A1B1C1G* is the same as that between a finite quantity, *A1B*, and an infinitely small quantity, *A0B*. Therefore, the rectangle *R* has an infinite dimension. Q.E.D.

Although in *DQAC* Leibniz appeals to infinitary concepts, he nevertheless does not apply the calculus and infinitesimal notation, which he was already developing at that time. Our second example comes precisely from the time when he had already developed the basic rules of the infinitesimal algorithm, which he finally published in 1684. As we have pointed out earlier, in that year Leibniz published for the first time the differentiation rules of the five operations in *Nova methodus pro maximis et minimis*. In this work, Leibniz introduces the differential notation, *dx*, for differential quantities and formulates the rules of operation with such quantities for the five fundamental operations: addition, subtraction, multiplication, division and radication (GM V 221-222; Leibniz, 1995, pp. 106-109). Infinitely small quantities are exhibited as infinitely small increments or differences between finite quantities, and its interpretation is mainly geometric, for example, in terms of infinitely small increments of abscissas and ordinates or of infinitely small sides of infinitangular polygons (GM V 223). Leibniz remarkably does not justify the rules of differentiation, but, after explaining their general properties, he just presents examples of their use to differentiate equations (GM V 223-224) and to give proof of the law of Snell on the relation of the angles of incidence and refraction of a light ray (GM V 224-225; Hess, 1986, p. 70). However, a few years earlier, Leibniz provided a justification for a preliminary version of these rules in an unfinished treatise entitled *Elementa calculi novi pro differentiis et summis, tangentibus et quadraturis, maximis et minimis, dimensionibus linearum, superficierum solidorum, aliisque communem calculum transcendentibus* (HOCD 32-38; Leibniz (1855), pp. 149-155; Hess (1986), pp. 97-102. English translation: Child (1920), pp. 134-144. We follow Hess' edition). The essay has not been dated, but according to Child it was written in the 1680s (Child 135) and Hess, its most recent editor, dates it before 1684 (1986, p. 71). Unlike *Nova methodus*, in *Elementa calculi* Leibniz not only introduces the differential notation *dx*, but also the integration operation $\int dx$, as well as the fundamental law of the calculus, which states that differentiation and integration are inverse operations (Hess, p. 100; Child, p. 142). Thus, Leibniz introduces a preliminary method for obtaining integrations from differentiations (Hess, p. 102). These are the differentiation rules that we find in *Elementa calculi* (Hess, pp. 101-102; Child, pp. 142-144):

Addition: $d(x+y) = dx + dy$
Subtraction: $d(x-y) = dx - dy$
Multiplication: $d(xy) = xdy + ydx$
Division: $d\left(\frac{x}{y}\right) = \frac{xdy - ydx}{x^2}$
Exponentiation: $d(x^e) = ex^{e-1}dx$
Radication: $d\sqrt[r]{x^h} = \frac{h}{r}\sqrt[r]{x^{h-r}}$

Except for the rules of addition and subtraction, the rest are all accompanied by their justification. We will reproduce here the demonstration of the multiplication rule, in which how Leibniz proceeds by "eliminating" differential quantities is clear:

Assuming *dx*, *dy* as "infinitely small" differential increments of *x* and *y*, we have that:

$$(1)\ d(xy) = (x + dx).(y + dy) - xy$$

If we develop (1), we obtain that

$$(2)\ d(xy) = xy + xdx + ydx + dxdy - xy\ \textbf{\textit{that is}}\ xdy + ydx + dxdy$$

Now, the fundamental step is to eliminate the product *dxdy*, which Leibniz justifies on the basis that it constitutes an infinitely small quantity not only in relation to the finite quantities *x* and *y*, but also in relation to *dx* and *dy*, *which are infinitely small quantities*. In other words, *dxdy* is an *infinitely infinitely* small quantity, and therefore its elimination is justified; consequently, we obtain that

$$(3)\ (dxy) = xdy + ydx$$

This kind of justification of the rule, which anticipates the introduction of infinitesimal quantities in terms of *relative* incommensurable quantities (GM VI 150; GM IV 91-92, see also Rabouin & Arthur, 2020, pp. 432-433) reappears in a letter from Leibniz to Wallis, where he appeals precisely to the concept of incomparable quantity to justify the elimination of *dxdy* (GM IV 63, A III 8, 92. Rabouin & Arthur, 2020, pp. 437-438). The justification for the division rule appeals to the same procedure of eliminating infinitely small quantities.

The cases we have considered were formulated in different periods of Leibniz's development of infinitesimal mathematics. In the first one, we have tried to show how Leibniz appeals to geometric infinitary fictions to prove a geometrical theorem , while in the second case he introduces a property of infinitely small fictional quantities, that is, "incommensurability", to justify the elimination of infinitely small quantities (in this case, *infinitely infinitely* small ones). At the moment, we are not concerned with *justifying* these infinitary procedures of Leibniz. For a discussion of this problem, we refer to the works of Bos (1974), Knobloch (2002), Rabouin (2015) and Rabouin & Arthur (2020).

5. Mathematical fictions and impossibility

It is now necessary to clarify the notion of mathematical fiction, with regard to the question of the existence or nonexistence of its corresponding objects.[8] If by "fiction" we understand a notion without a denotation, we run the risk of throwing the baby out with the bathwater, because Leibniz recognizes that mathematical objects are ideal in nature, having an incomplete nature and therefore they lack real or substantial existence.[9] Thus, we could

[8] Regarding this question, it is worth mentioning that Levey (2008, pp. 123-128) exhibited three senses in which fictions can be conceived, without trying to unravel Leibniz's own notion, but anachronistically, based on three ways in which scientific theories can be interpreted. Thus, he distinguished (1) "reductionism", according to which Leibniz's infinitesimal language can be reduced to a language that includes only finite terms (that is, the syncategorematic interpretation to which the author subscribes); (2) "pragmatism", according to which the infinitesimal language is an adequate way, scientifically speaking, to describe the data that the theory attempts to organize, explain and predict; and (3) "ideal-theory instrumentalism", according to which an infinitesimal is a device "for inferring meaningful results from meaningful premises" (p. 124).

[9] Leibniz seems to have developed this conception of mathematics as something "ideal" at the end of his Parisian period and especially in the 1680s. Cf. A II 2 75; A VI 4 991; GP 4 490, 561; GP 2 225/OFC 16B

say that they are notional or "thought" objects, but that as such they cannot be found in factual reality, nor do they have an existence in themselves, independent of their being thought or conceived.

Now, such a conception, which Leibniz emphatically holds in his mature thought, raises the question as to whether the concept of fiction as a notion without a denotation (beyond its confusion) could not be applied to the entire domain of mathematics. This consequence would be undesirable for our purpose, since it would no longer make much sense to try to clarify the place of fictions within mathematics, as opposed to "true" or "real" notions, to which some kind of denotation should correspond. Thus, the answer to this question is the same as clarifying the Leibnizian concept of mathematical existence. In other words, if we can properly elucidate what it means for Leibniz that a mathematical object exists, we will have a secure basis for distinguishing "true" or "real" mathematical notions from merely fictitious and instrumental ones.

In relation to this, the Leibnizian answer to the problem of the existence of mathematical objects seems clear: in mathematics, the criterion of existence is possibility. In other words, returning to some previous considerations, a mathematical object is admissible if it is possible, and the feature and fundamental criterion of possibility is consistency. In other words, for a mathematical object to exist, it is enough to prove that a contradiction does not follow from its concept. This existence cannot of course be assimilated to the way in which physical objects exist, since it has a notional character in the sense of an existence-in-idea or "in thought" (moreover, they are possible objects in the mind of God).[10] In conclusion, the existence of mathematical objects means straightforwardly to be possible. In *De libertate et necessitate*, Leibniz precisely points out the relationship between mathematical existence and possibility:

> I therefore say that it is possible that of which the essence or reality is something, that is, which can be distinctly thought. (A VI 4, 1447)

In other words, a notion is possible if its notional components, when separately thought, are compatible, that is, they do not imply a contradiction. From the fact that in nature there is not, nor has there been, nor will there be a perfect geometric figure (for instance), it does not follow that this figure is neither possible nor thinkable. That is to say, for something to have an essence means, as Leibniz maintains, that it has an "ideal" or "conceptual" existence or being. Thus, for example, although there is no perfectly circular object in nature and we cannot form an image of a perfect circle because of our cognitive limitations (as Leibniz explained in *Numeri infiniti*, A VI 3, 498-499), the circle is something possible, that is, there is an essence of the circle and thus it is thinkable (A VI 3, 463). The same could be said, for example, of a pentagon: the fact that in nature there has not been and will not be an exact pentagon (*accuratum pentagonum*; A VI 4, 1447) does not make it less possible. Just like in the case of the circle, we can give a definition of a pentagon and ultimately we can think and demonstrate things about this figure. Thus, mathematical entities in general are possible without an existence outside our minds.

Hence, as anticipated the distrust about concepts that can hide contradictions and therefore have no denotation, entails the need to find proofs of possibility based on a

1164, inter alia. For the question of the origins of the "ideal" conception of mathematical entities, see Esquisabel and Raffo Quintana (2020).

[10] It is outside the scope of this paper to explain the difficult connections between our thoughts and those of the divine mind.

demonstration of consistency. Likewise, the difficulties involved in analyzing consistency in geometry provide the basis for the acceptance of the method for the proof of mathematical existence based on genetic or causal definitions, which basically consist of a constructive norm (this question is closely connected with the importance that Leibniz gives to causal definitions as proof of possibility. Cf. A VI 4 542-543; A VI 4 589-590; GP 2 225, *inter alia*).

In conclusion, if mathematical existence is mere possibility, the difference between a "real" notion and a fiction in the mathematical domain seems to be given by the possibility or impossibility of the corresponding object, which, at first glance, must be determined by means of a consistency test. In other words, it seems that we must conclude that mathematical fictions are notions of impossible and therefore self-contradictory objects. In this way, whenever we have a mathematical fiction, we should be able to show its inconsistency. Thus, for example, in relation to our main topic, we should be able to show the self-contradictory character of all infinitary concepts, whether they are infinite numbers, infinite bounded lines or infinitely small quantities.

However, an examination of Leibniz's arguments concerning the fictionality of infinitary objects reveals a more prudent attitude. For example, unlike the self-contradictory character of the infinite number, Leibniz's arguments for rejecting infinitary objects such as bounded infinite lines or infinitely small quantities are often more nuanced, in the sense that they do not point out the contradiction, but rather conclude paradoxical or inadmissible properties, which make their existence at least "improbable" (for a discussion of Leibniz's arguments regarding the contradictoriness of the infinite number, see Esquisabel and Raffo Quintana 2017; Fazio 2016, pp. 164-169; Brown 1998, pp. 113-125; Brown 2000, pp. 21-51; Levey 1998, pp. 49-96 and Lison 2006, pp. 197-208).

This suggests the idea that for Leibniz both the existence and the mathematical possibility or impossibility are not a matter of absolute oppositions but they do to some extent admit degrees. In this vein, for example, Lison (2020, p. 281) pointed out that there are some mathematical quantities that, although they cannot be clearly and distinctly perceived, "can be considered to be imaginary in the sense that they belong to the ideal scope of mathematics", since "they do not include a contradiction (otherwise they would be excluded from being possibilities) but neither are they candidates for actual realization". From this perspective, we argue that different concepts of mathematical possibility and impossibility can be detected in Leibniz's work and hence different degrees and hierarchies of fictionality should also be defined. This observation constitutes another argument to maintain that the problem of possibility in terms of consistency is a starting point for dealing with the question of mathematical existence, but it does not exhaust it. Thus, there are valid reasons to introduce the thesis that, since the mathematical existence is closely related to the possibility, it will ultimately depend on the different gradations that Leibniz admits for this concept. As we shall see, what is possible includes what is relatively possible or *secundum quid*, that is, what is possible in relation to certain principles or points of view. In conclusion, we can distinguish different concepts of impossibility, ranging from the most rigorous one, based on inconsistency, to other looser ones. These different forms of impossibility also force us to recognize different forms of mathematical fiction, as we will see in what follows.

6. Three concepts of mathematical possibility and impossibility

Thus, in Leibniz's considerations on the nature of mathematical objects, three concepts of possibility and correlatively of impossibility can be distinguished, namely: absolute possibility/impossibility as consistency/inconsistency; (relative) possibility/impossibility as mathematical representability/irrepresentability; and (relative) possibility/impossibility as compatibility/incompatibility with architectonic principles of order of the world, such as, for example, the principle of continuity and of sufficient reason.[11]

We have already discussed the first case of possibility/impossibility. Absolute possibility is indeed given by the absence of contradiction. In other words, something is possible if its notion contains no inconsistency. Thus, for example, the number two and the circle are possible. As we saw, the consistency test is one of the main reasons for the Leibnizian preference for causal or "constructive" definitions. Correlatively, the impossibility is given by self-contradiction or conceptual inconsistency, as occurs with the notion of a 'square circle' or of 'the number of all numbers'.[12]

However, Leibniz recognizes other forms of possibility/impossibility. A second class of this pair of modal notions is given by the possibility of providing some kind of instantiation or geometric representation in the proper sense of the word (not an analogical one) to a mathematical concept. For example, the infinitely small abscissa of our first example can be represented only analogically by a finite abscissa. Moreover, the fact that something is geometrically irrepresentable also implies that the conditions for solving the problem or for the construction of the corresponding entity are not given. In that case, that which is geometrically representable is possible, such as finite magnitudes or the "real" roots of an equation, while that which is geometrically irrepresentable is impossible, namely, imaginary roots and infinitely small or infinitely large bounded quantities.

Finally, some meditations of Leibniz indicate that he conceived of a third type of possibility/impossibility pairing, which arises from the compatibility or incompatibility with the architectonic or rationality principles that govern the constitution of the world, such as the principle of continuity and the principle of sufficient reason. In that case, what adapts to and is compatible with these architectonic principles is possible. Thus, for example, mathematical continuity, unbounded magnitudes and potential infinity fulfill the requirements of sufficient reason and the law of continuity or order, and hence they are "cosmologically" possible, that is, admissible within the structural order of a rationally organized world. On the other hand, notions of objects that violate or are incompatible with those same principles of organization of the world are impossible. This is the case with infinite concepts such as infinitely small or infinitely large bounded quantities. As we shall see, the admission of both concepts effectively constitutes a violation of the principle of

[11] It may be surprising to say that infinitely small quantities could be incompatible with the principle of continuity. But we take that principle here in the sense of a principle of order, that is, nature should be orderly constructed (see, for example, A VI 3, 564-565, and GP II 193; 282). Thus, to suppose the existence of infinitely small real quantities could entail the thought that motion would *really* be composed of infinitely small jumps in infinitely small parts of space and time, and *this* goes against the order of nature.

[12] In accordance with this and in an illustrative way, in the Parisian period Leibniz noted: "It is not admirable that the number of all numbers, all possibilities, all relations, that is, reflections, are not distinctly intelligible; in effect, they are imaginary and do not have anything that corresponds in reality [*a parte rei*]" (A VI 3, 399). However, there is also a difference between what is manifested in this passage and what he does later. For, in this passage epistemic elements prevail for possibility and impossibility, expressed in the fact that they "are not distinctly intelligible", while the "logical" criterion based on consistency, which already appears in the Parisian period, is however much more clearly formulated after this period.

sufficient reason and of continuity (for the application of architectonic principles in Leibnizian philosophy, especially in the construction of its dynamics, cf. Duchesnau 1993, ch. 4; Duchesnau 1994 and Duchesnau 2019, pp. 39-62; in relation to the principles of sufficient reason and continuity, see Nicolás 1993 and Luna Alcoba 1996).

The distinction between possibility/impossibility as consistency/inconsistency and possibility/impossibility as representability/irrepresentability appears in many Leibnizian texts from his youth and also from the 1680s. It is introduced as a difference between impossibility in the strict or essential sense and *per accident* impossibility. The fact that Leibniz introduces the distinction based on analogies with mathematical objects is central to our purposes. Thus, Leibniz introduces an analogy between impossibility of essence and contradictory mathematical notions, while the impossibility of existence is analogous to geometric irrepresentability.

Thus, for example, in the *Confessio philosophi* (1672-1673), Leibniz argues that, rigorously speaking, impossibility implies unintelligibility, despite the fact that, in a broad sense, we can call *per accident* impossible that which, being intelligible, that is, possible in a proper sense, is impossible from the point of view of existence (in some sense of the word). Thus, for example, he points out: “therefore, the necessity and impossibility of things are to be sought in the ideas of those very things themselves, not outside those things. It is to be sought by examining whether they can be conceived or whether instead they imply a contradiction” (A VI 3, 128. Translation: Leibniz 2005, 57). It is important to highlight the relation between the notions of impossibility, possibility and intelligibility, especially insofar as the latter is a condition of possibility. In other words, a thing can be possible, that is, intelligible, despite the fact that it “cannot” ever exist and is thus “impossible” in an improper sense:

> Therefore if the essence of a thing can be conceived, provided that it is conceived clearly and distinctly (e.g., *a species of animal with an uneven number of feet, also a species of immortal beast*), then it must already be held to be possible, and its contrary will not be necessary, even if its existence may be contrary to the harmony of things and the existence of God, and consequently it never will actually exist, but it will remain *per accidens* impossible. Hence all those who call impossible (absolutely, i.e., per se) whatever neither was nor is nor will be are mistaken. (A VI 3, 128. Translation: Leibniz 2005, 57)

Thus, the fact that something is *per accident* impossible does not imply that it is absolutely impossible. In other words, what is intelligible but has not existed, nor does exist, nor will exist, is not however essentially impossible. At the end of the Parisian period, Leibniz returned to the question of the impossibility from a very similar approach to that of *Confessio philosophi*, although this time emphasizing the analogies which can be established between the question of the possible and the impossible with the domain of mathematics. As we will see, this will be a constant in the following years. Two texts from the end of 1675 give a clear account on this, namely *Imaginariae usus ad comparationem circuli et hyperbolae* of November 29, 1675 and *De mente, de universo, de Deo*, from mid-December of the same year. Unlike the *Confessio philosophi*, Leibniz does not repeat the distinction between absolute impossibility and *per accident* impossibility, but argues that “impossible” is a two-fold notion: on the one hand, what has no essence is impossible and, on the other, that which lacks existence because it is inharmonious is also impossible:

> In the same way, there is a two-fold reason for impossible problems: one, when they are analyzed into a contradictory equation, and the other, when there is an analysis into an imaginary quantity, for which no place can be understood. (A VI 3, 464. Translation: Leibniz (1992, 7))

Thus, Leibniz establishes an analogy such that the impossibility of essence corresponds in mathematics with the problems that are solved in contradictory equations, such as 3 = 4, while the impossibility of existence corresponds in mathematics with the problems that are solved in quantities that are geometrically irrepresentable. In a later text, *De libertate et necessitate*, written between 1680 and 1684 (A VI 4, 1444-1449), Leibniz returns to this point and illustrates the analogy in greater detail. In order to show the difference between the possibility of essence and the impossibility of existence, Leibniz once again appeals to examples based on solving equations. In this way, Leibniz goes back to the difference between two kinds of unsolvable problems, that is, those that are solved in a contradiction and those that are resolved in a quantity that *cannot* numerically *be designated*. For the first case he gives the example of solving for the unknown quantity in the system of equations $x^2 = 9$ and $x + 5 = 9$, which gives an impossible result, namely 3 = 4. For the second one, he give the example of the equation $x^2 + 9 = 3x$, which has roots with an imaginary component. Although this is not a contradictory equation, it is solved in a quantity which cannot be exhibited, that is, such that the corresponding geometric construction cannot be assigned to it (A VI 4, 1448). Precisely, regarding the analogy between the impossibility of existence and imaginary quantities, Leibniz states:

> (…) this [scl. the impossibility of existence] can be optimally illustrated in the likeness [*similitudine*] of imaginary roots in Algebra, since $\sqrt{-1}$ involves some notion, although it cannot be exhibited. Indeed, if someone wants to exhibit it in a circle, then he or she would find that that circle is not touched by the line required for it. (A VI 4, 1448)

The correlation between factual existence and geometric representability, as well as between nonexistence and irrepresentability, is complex and must be taken within the limits of the analogy, especially because it is not necessarily true that something should be considered to be irrepresentable if it does not strictly correspond with reality. However, the analogy is based on the fact that there are quantities that cannot be geometrically represented, namely imaginary quantities. As is known, Leibniz recurrently established a connection between the question of fictitious quantities and imaginary roots (as examples of his mature thought, A VI 4, 1448; *Leibniz to Foucher*, A II 2, 495-496; *Leibniz to Bernoulli*; A III 7 796-797; *Leibniz to Varignon*, GM 4, 98; *Leibniz to Hermann*, A III 9, 469; *Leibniz to Grandi*, GM 4, 218-219; regarding this issue, cf. Sherry & Katz 2012). In connection with this, the features of the imaginary roots that can be extracted from the following passage of *Imaginariae usus ad comparationem circuli et hyperbolae* are very significant for our purposes:

> I don't know what needs to be discussed more diligently than whether the quantity $\sqrt{-1}$ is nothing at all or if it actually contains [something]. Although it cannot be carried out, however it can be understood, not by itself, but with the help of characters and analogy, such as, for example, those thoughts that I call blind. And indeed, just as there are incommensurable [quantities] that are powers of commensurable [quantities], so there are also imaginaries ones whose powers are real [quantities]; that is, there are impossible [quantities] whose squares are possible, such as, for example, $\sqrt{-1}$ whose

> square is -1, even if it is claimed that there is nothing at all in nature that corresponds to such a quantity; however, it is enough for its character to be useful, since it expresses real things when [the character] is joined with other things. (A VII 7 [draft version], 221)

We can point out two main groups of features of imaginary roots worth noting here, namely, those related to the question of possibility and impossibility, on the one hand, and those related to their use for mathematical practice, on the other. In the first place, imaginary roots are impossible quantities in the broad sense of the expression, that is, they cannot be carried out (*non possit effici*), that is, they cannot be represented geometrically. Secondly, these roots are operationally thinkable with the help of characters and analogy: although these quantities are not possible, we take them *as if* they were when we operate with them, that is, when we give effective rules of operation, as we have pointed out in point 4 of the introduction. In turn, as we anticipated in point 5 and in the section dedicated to symbolic knowledge, characters constitute an indispensable support because in context, that is, in the procedures we carry out using these impossible quantities together with other possible quantities we can solve problems and thus “express real things”. There are additional problems in relation to imaginary quantities, which for the sake of order we will consider in the next section. For the rest, in what exact sense this latter condition must be understood is precisely one of the pending tasks in the analysis of mathematical fictions.

Unlike *De libertate et necessitate*, in *Elementa nova matheseos universalis* (1683), the distinction between equations with absurd resolutions and the geometric irrepresentability of imaginary solutions is not introduced as a mere analogy to illustrate the difference between the impossibility of essence and the impossibility of existence, but rather this distinction truly acquires mathematical relevance. In this text Leibniz expressly applies the difference between absolute impossibility and *per accidens* impossibility to mathematical concepts. The importance of the passage justifies the fact that we quote it in full:

> There is a big difference between imaginary or accidentally impossible quantities and absolutely impossible quantities, which involve contradiction, as when it is found that for solving the problem it is necessary for 3 to be equal to 4, which is absurd. However, imaginary quantities, that is to say, impossible by accident, namely, those that cannot be exhibited because of a lack of sufficient constitution, which is necessary for the intersection, can be compared with infinite or infinitely small quantities, which arise in the same way. (A VI 4, 521; Leibniz 2018, 108)

As we can see, Leibniz takes up the question of imaginary quantities almost in the same way: the square roots of negative numbers, geometrically interpreted, represent an intersection that does not occur between a line and a circle. Although the distinction between absolute impossibility and *per accidens* impossibility is based on basically the same examples as in *De mente, de universe, Deo* and *De libertate et necessitate*, it is central to our approach that the context of application is a strictly mathematical one. The inclusion of infinitary concepts, such as infinitely small or infinitely large quantities, within the class of accidentally impossible notions is equally important. In that case, Leibniz’s used the examples of the right angle, understood as an angle that has an infinitesimal difference in regards to the perpendicular, and parallel lines with an intersection at infinity. In addition, Leibniz adds that, although for an inexperienced person in mathematics these fictions seem to lead to absurd conclusions, they are not only productive for the calculation, but also, in their practice, we are necessarily led to them (A VI 4, 521).

Finally, the impossibility as incompatibility with architectonic principles appears more than once throughout the different periods of Leibniz's thought. Thus, in *De mente, de universo, de Deo*, the impossibility of existence is based on the incompatibility with divine reasons for things to be or not to be (A VI 3 464-465). A similar consideration but applied to mathematical objects, appears in *De libertate et necessitate*, a text to which we have previously referred. As we saw, Leibniz returns to the distinction between the possibility of essence and the possibility of existence and illustrates it in the following way:

> For example, even if we imagine that in nature no exact pentagon has existed and never will exist, nevertheless the pentagon would still be possible. However, a reason must be given as to why the pentagon did not exist or will never exist. There is no other reason for this situation than the fact that the pentagon is incompatible with other things that include a greater perfection, that is, that involve more reality, so that they will certainly exist instead of it. Now, if it is inferred that for this reason it is necessary that the same pentagon does not exist, I concede the conclusion, if its meaning is that the proposition "the pentagon will not exist nor has it existed" is necessary, but it is false if its meaning is that the proposition "no pentagon exists" (which makes abstraction of time) is necessary. I indeed deny that this proposition can be demonstrated, *since the pentagon is not absolutely impossible and does not imply a contradiction, although it follows from the harmony of things that it cannot find a place in things*. (A VI 4 1447-1448; emphasis added)

The passage without doubt contains more questions than we can deal with here. However, we would like to highlight three central assertions for our interpretation: in the first place, Leibniz clearly determines that not every single thing that is possible absolutely speaking, that is to say, that is non-contradictory, is also possible from the point of view of real existence; secondly, the impossibility relative to the real existence or "the series of things" obeys reasons of perfection and harmony, that is to say, reasons of order; last, but not least, there is a way of understanding existence that puts aside considerations of time and place, that is, the close connection of the existence of mathematical objects (the pentagon) with possibility reappears.

In any case, Leibniz's argument aims to show the distance that exists between the "pure" or "absolute" possibility and the real possibility, relative to the existence of the most perfect and harmonic series of things. The requirement of compatibility, harmony and order is generally Leibniz's argument against the real existence of infinitary objects. As we have seen in the case of *De mente, de universo, de Deo*, the issue at stake is the concordance of existence with divine reason, that is, the grounds that God finds for something to be or not to be. It is not uncommon to find arguments of this kind to reject the existence of infinitary objects in Leibniz's writings of the Parisian period. Thus, for example, in the *Pacidius Philalethi* Leibniz at least twice states his rejection of infinitary objects, based on the principle of sufficient reason.

Firstly, while it does not directly concern the question of the existence of infinitesimals, although it implies it tangentially, Leibniz presents an argument that appeals to reasons of harmony and congruence. The context of the discussion consists of the explanation of motion in terms of the annihilation of a body in one position and its recreation in the next one. Thus, motion can be understood as an infinitely small "leap" from one position to the next, through an act of destruction and creation (A VI 3 560).

Leibniz's refutation of this conception of motion appeals to the principle of sufficient reason, through arguments based on harmony and congruence; indeed, "(…) this opinion (…) is offensive to the beauty of things and the wisdom of God" (A VI 3 560. Translation: Leibniz 2001, p. 199), because:

> (…) the supremely wise author of things does nothing without a reason; yet there is no reason why these miraculous leaps should be ascribed to this rather than that grade of corpuscles (…) (A VI 3 561. Translation: Leibniz (2001, p. 199)).

Although Leibniz's argument exhibits other facets which we will not develop here (for an analysis we refer to Raffo Quintana 2019, 60-61, and Esquisabel and Raffo Quintana 2020), it can be synthesized in the thesis that accepting a break in the analysis of the motion of a body by introducing a "minimum last leap" would constitute a violation of the uniformity of the motion, implying that for each traveled path there is a smaller one *ad infinitum*. In conclusion, there would not be a sufficient reason to accept such leaps.

Be that as it may, a subsequent examination of the nature of motion introduces the consideration of infinitely small lines and times, precisely in connection with the possibility of resuming the explanation of the change of place by leaps through infinitely small spaces and times. The development of this hypothesis would imply the existence of infinitely small spaces and times (A VI 3 564). The response of Pacidius, Leibniz's *alias* in the dialogue, is a categorical rejection of this possibility (Raffo Quintana 2019, p. 81):

> 1 would indeed admit these infinitely small spaces and times in geometry, for the sake of invention, even if they are imaginary. But I am not sure whether they can be admitted in nature. For there seem to arise from them infinite straight lines bounded at both ends, as I will show at another time; which is absurd. Besides, since further infinitely small spaces and times can also be assumed, each smaller than the last to infinity, again no reason can be provided why some should be assumed rather than others; but nothing happens without a reason. (A VI 3 564-565. Translation: Leibniz 2001, p. 207).

Leibniz's conclusion adds an additional consideration to the rejection of infinitesimal quantities based on the violation of the principle of sufficient reason. In accordance with the argument synthesized some paragraphs before, the admission of infinitely small quantities would amount to violating the uniformity of nature, since we would have to admit the existence of quantities smaller than any others and there would be no reason for it. However, in addition to the transgression of the principle of sufficient reason, Leibniz alleges another reason for the rejection: the existence of infinitely small lines would also imply the existence of infinite lines bounded at both sides, and Leibniz rejects this as "absurd". Nevertheless, as we saw in the section devoted to examples of infinite entities, despite the fact that he rejects their real existence, Leibniz employs them as mathematical fictions to solve mathematical problems. Once again, the admission of these fictions is justified by its usefulness for mathematical invention.

The reasons for denying real existence to this class of mathematical fictions are not circumstantial or momentary. In the discussion with Johann Bernoulli about the reality of infinitesimals, Leibniz goes against the real existence of infinitesimals in his letter of June 7/17, 1698 with the same argument as before: the existence of infinitely small quantities would imply the admission of bounded infinite lines, which implies absurd consequences such as the existence of a bounded time, that is, infinite but endowed with extremes:

> If we establish infinitely small real lines, it would follow that lines bounded on both sides would also have to be established, which, however, would be to our ordinary lines as the infinite to the finite; and since from this it would follow that exists a point in space which could never be reached in an assignable time by means of a constant motion; likewise, a time bounded by both sides would necessarily be conceived, which, however, would be infinite in such a way that it would happens, so to speak, as a kind of bounded eternity; or one could live without ever being possible to assign a bounded number of years to die and yet one day would die; for this reason, unless I am forced by incontestable demonstrations, I do not dare to admit all this. (GM III 499-500).

Many of the arguments supporting these paradoxical consequences –a limit that cannot be reached in a finite time, an eternity with extremes, an infinite but equally mortal life– can be found in writings that go back to the Parisian period and that according to the editors of the Academy edition belong to the *De summa rerum* project (for example, *De infinito observatio notabilis*, A VI 3 481). Although the examination of the arguments used by Leibniz in these writings exceeds the scope of this paper (cf. Esquisabel and Raffo Quintana 2020), these absurd consequences, though not strictly contradictory, are what motivate Leibniz to reject the real existence of infinitary objects and sustain their impossibility in terms of incompatibility with the nature of an orderly and harmonious world. As he responds to Johann Bernoulli in his letter of November 18/28, 1698,

> As regards infinitesimal terms, it seems to me that not only we cannot reach them, but that they do not even exist in nature, that is, they are not possible; otherwise, as I said, if I admitted that they are possible [*esse posse*], I would concede that they exist. Therefore, it would be necessary to see under what reason it can be shown that it is possible, for example, an infinite straight line, but bounded by both sides. (GM III 551).

Leibniz’s statement to Bernoulli regarding the question of the existence of the infinitely small and the infinitely large could not be more illustrative as a sign of Leibniz’s prudence: Leibniz certainly does not uphold the categorical impossibility of the existence in nature for infinitary objects, but limits himself to a weaker claim: up to now, the possibility of their existence has not been demonstrated. In other words, he only claims the presumption of its impossibility until the contrary is proven, in which case he would be willing to admit its existence. Thus, if there is only a presumption of impossibility, based on the reasons we have earlier proposed (absurd consequences, inconsistencies, violation of the principle of sufficient reason), apparently Leibniz would not be willing to hold that there is a categorical proof of impossibility founded on self-contradiction either.

From this perspective, it is permissible to maintain that Leibniz only presumptively rejects the existence of infinitary objects based on architectonic reasons; for that reason, we also assign them an impossibility of the third type, namely, due to incompatibility with the principles that articulate actual reality. The presumptive nature of this impossibility and the provisional rejection of the existence in nature of infinitary objects explains in some way Leibniz’s prudent attitude regarding the possibility of resolving or not this metaphysical question, as well as his recommendation, reiterated each time he faces this metaphysical problem, of remaining within the realm of mathematics, where the admission of the infinitely small and the infinitely large are justified by their methodological effectiveness, as he maintains in *Cum prodiisset*:

> Meanwhile, I confess that it can be doubted whether this state of momentary transition from inequality to equality, from movement to rest, from convergence to parallelism, or the like can be sustained in a rigorous and metaphysical sense, that is, it can be doubted that infinite extensions, some greater than others, or the infinitely small ones, some smaller than others, are real. And whoever wants to dispute about these questions will find him embroiled in metaphysical controversies about the composition of the continuum, on which there is no need for geometric questions to depend. (*HOCD*, p. 43)

7. The reconsidered concept of mathematical fiction

At this point, we will try to connect the lines of argumentation that we have developed so far with the considerations we proposed for the notion of mathematical fiction. We initially analyzed mathematical fiction as a symbolic notion devoid of denotation or reference, the latter consisting of an idea or better an "ideated" or "in-idea" object. Now, we have elucidated the lack of denotation in terms of the impossibility of existence of the corresponding object, and, with regards to mathematical fictions, we preliminarily understood such impossibility in terms of inconsistency. However, the analysis of Leibniz's texts has revealed the existence of three kinds of impossibility, in such a way that, according to this result, it is necessary to expand the concept of mathematical fiction considered as a starting point. Consequently, three concepts of mathematical fiction can be distinguished, in correspondence with the expansion of the concept of impossibility.

Firstly, the concept of absolute impossibility based on inconsistency corresponds to fiction$_1$, which delimits the class of inconsistent mathematical notions, such as the concept of "number of all numbers", which we repeatedly mentioned. Secondly, the concept of impossibility due to geometric irrepresentability corresponds to fiction$_2$, which groups together mathematical notions that cannot be geometrically instantiated or exhibited, such as imaginary roots, the extremes of infinite lines, the common points of parallel lines to each other and infinitely small quantities. Finally, the third kind of fiction –fiction$_3$– arises out of the concept of impossibility due to incompatibility with architectonic principles, which once again affects the "infinitary" concepts. In regards to fiction$_3$, it should be finally added that infinitary fictions are of a "presumptive" nature, in the sense that their objects are considered impossible until their possibility is proven.

As we have anticipated at the beginning of our inquiries, Leibniz applies in mathematics, in one way or another, the three classes of fiction, whether considering the case of infinite wholes (the case of infinite series), imaginary roots or infinitary objects. From the point of view of symbolic knowledge, the introduction of fictions can take place through verbal or written discourse, that is, using common language terms whose meanings can be clarified in the best-case scenario by a merely nominal definition, such as "the infinite number is the number of all numbers" or "an infinitely small quantity is a quantity lesser than any assignable quantity",[13] and so on. Another way of representing a

[13] Actually, Leibniz appeals to various ways of referring to infinitesimal or infinitely small quantities: "quantity smaller than any assignable quantity" is one of them, but there is a plurality of characterizations that are only apparently equivalent. Moreover, it can be shown that there is an evolution in the way that Leibniz characterizes infinitely small quantities. Although we cannot develop it here and we will do so in a later study, we maintain that the different ways of designating or characterizing infinitely small quantities denote an evolution in the way that Leibniz conceived of the mathematical function of such fictions.

mathematical fiction appeals to an integrable and manipulable symbolic element in the context of a calculus, as in the case of differential quantities, and the same could be said of the series given by infinite expansion, with which Leibniz operates as if they were given infinite totalities of terms. In particular, the algebraic symbols that represent infinitary quantities constitute an outstanding case of a blind symbolic notion, since in themselves they lack denotation, and hence their introduction is operationally justified in the sense that they allow us to efficiently solve mathematical problems, such as quadratures or problems about minima and maxima. It must be added to this that the arithmetic operations with infinitary quantities (addition, subtraction, product, powers and radication) constitute an analogical extension of the operations with finite quantities to (fictional) infinitary objects. Finally, a third class of representation of infinite quantities is the geometric or diagrammatic representation, which also exhibits a markedly semiotic character. In that case, the representation (when it can be carried out) is purely analogical, since an infinite quantity is in itself irrepresentable or non-instantiable. This is the case, for example, of infinitesimal quantities such as differential increments of abscissa and ordinate or rectangles with infinitely large and infinitely small sides. In all these examples, the diagram allows us to operate with infinite or infinitesimal quantities as if they were finite or as if they had properties analogous to those of finite quantities. In this regard, E. Grosholz has developed an interpretation of this analogical use of finite quantities in terms of a theory of ambiguous signs, which, when representing finite quantities, are valid as iconic signs, while, when they represent infinitary quantities, they work as symbols in the Peircian sense of the term (Grosholz 2007, ch. 8).

There is a particular case of mathematical fiction that seems to contradict our interpretation of mathematical fictions, namely, the case of imaginary quantities. Leibniz proposes a method for providing real roots in terms of "in appearance" (*in speciem*) imaginary quantities (GM VII 141-144), when dealing with the *casus irreducibilis* of the cubic equation, for which the real solutions of the equation can only be expressed through imaginary values. These are "in appearance" imaginary quantities precisely because we employed them to express real roots that cannot be obtained by other algebraic methods. Leibniz introduces the concept of an "in appearance imaginary quantity" in *De resolutionibus aequationum cubicarum triradicalum, de radicibus realibus, quae interventu imaginariarium exprimuntur, deque sexta quadam operationem arithmetica* (GM VII 138-153), a text in which the study of a general method for solving cubic equations is considered, and these quantities are also mentioned in *Elementa nova matheseos universalis* as "in appearance impossible" quantities (A VI 4, 520; cf. Leibniz 2018, pp. 107-108).

Thus, the concept of an "in appearance imaginary quantity" or of an "in appearance impossible quantity" seems to imply an objection against our interpretation, for these quantities seem ultimately to refer to real quantities, since imaginary quantities are put in equivalence with real quantities. However, in our view in appearance imaginary quantities do not conflict with our general interpretation of mathematical fictions. For, as we have pointed out at the beginning of our work, the introduction of mathematical fictions contributes to the creation of new procedures and methods that expand or complete previously existing methods or theories, especially if the introduction of these fictions can be carried out by clearly formulated rules of symbolic operation. In our interpretation, this is the case of in appearance imaginary quantities. Leibniz justifies expressing real roots in terms of imaginary quantities because in this way the method of solving cubic equations by the reduction to a quadratic equation is coherently completed, thus providing generality while lying within the field of "algebraic" or "analytical" solving methods, as clearly emerges from his critique of the Cartesian method, which appeals to a geometric procedure. In regard to the Cartesian method, Leibniz holds:

> For analytical notations are of one kind and geometric ones are of another kind. The first ones are used to enunciate an unknown quantity in relation to certain arithmetic operations, such as additions, subtractions, multiplications, divisions, roots extractions, and transformations [*reformationes*] of imaginary quantities (which I added to the previous ones). The second ones, however, enunciate an unknown quantity in relation to some geometric operations and the drawing of lines. (GM VII 143-144)

Ultimately, the rejection of the Cartesian solution, however clear it may be, is based on the fact that it is a geometric construction, while a strictly algebraic solution must be suitable for calculation or "algorithmic" procedures:

> It is characteristic of the analyst to express unknown quantities using always notations that are suitable for the calculation. However, it is clear that Descartes' notation, with which unknown quantities are expressed through relations with arcs, is not useful for calculating or, if calculates, it always does it with an invariable measure. (GM VII 144)

From this point of view, it seems clear that Leibniz considers the introduction of "in appearance" imaginary quantities as a syntactic procedure of a symbolic nature, regardless of the fact that a geometric interpretation of it can be given or not. In this way, the "purity" of the algebraic method remains, while at the same time it provides completeness. This conclusion is clear when Leibniz refers to the expression of real roots through imaginary quantities in terms of a new *operation*, which Leibniz calls "transformation of imaginaries in terms of reals" (*reformatio imaginarios ad reales*). This operation, the details of which are beyond the scope of this paper, allows us to express the sum of two real roots $x + y$ in terms of imaginary quantities (GM VII 140), making thus sense of the fact that x and y are expressed by imaginary quantities (GM VII 141). Thus, Leibniz concludes:

> And it is worth noting that we will have a *sixth kind of operation*, be it arithmetic or analytical one. For besides addition, subtraction, multiplication, division and roots extraction, we shall have the transformation [*reformatio*], that is, the reduction [*reductio*] of imaginary expressions to real ones. Addition and subtraction certainly reduce what is compound to the simple, that is, the parts to the whole or conversely, while multiplication reduces causes to effects, just as division and roots extraction make the reverse path. Finally, the transformation [*reformatio*] reduces imaginaries to reals. (GM VII 141)

In summary, the concept of in appearance imaginary quantity refers to a syntactic or purely "symbolic" operation, which is the transformation of imaginaries into reals. By virtue of this operation a real quantity is validly expressed by imaginary quantities. Thanks to this, the root system of cubic equation is completed and at the same time the generality of the method proposed by Tartaglia and Cardano for its solution is assured.

According to what we have proposed, Leibniz considers that infinite fictions are at the same time of the kind 2 and 3, that is, they are irrepresentable in the proper sense of the expression and also incompatible with architectonic principles. In other words, unlike the infinite number or the number of all numbers, for Leibniz infinitary concepts do not imply any contradiction, although they may imply paradoxical consequences, such as those already mentioned. As we pointed out earlier, it is true that in some texts, as for example in *Numeri infiniti*, a contradiction seems to be derived from the existence of infinitely small quantities, since the acceptance of the sum of bounded infinite series, endowed with a last infinitesimal term, seems to imply the existence of an infinite number, which, as we already know, is a contradictory notion (A VI 3 502-503). However, it is not a consequence that Leibniz himself categorically enunciates.[14] Whereas he constantly and consistently appeals to the "presumptive impossibility" argument based on considerations of incompatibility with principles, especially in his mature thought.

In any case, there is a question that requires clarification concerning the inclusion of infinitary notions such as fictions$_2$ and fictions$_3$. If infinitary quantities are fictions of both the second and third kind, this question naturally arises: why does Leibniz reject the existence of this kind of object from two different points of view, when in fact it would be enough with one kind of impossibility, be it the second or the third one? The answer to this question would probably require an analysis that goes beyond the scope of this work, and hence we will limit ourselves to giving only its general guidelines.

If we pay attention to the development of the problem of the fictionality of infinitary objects through the various phases of Leibniz's philosophy, we can see that arguments based on incompatibility with architectonic principles prevail. This insistence seems to indicate that Leibniz's preoccupation with the fictionality of infinitary entities is connected preponderantly to the problem of the real existence of infinite quantities, and not to that of their mathematical existence in terms of mere mathematical objects. In this regard, the progressive separation that Leibniz establishes between the field of mathematics, which restricted itself to ideal existence, and the domain of the actual existence of complete and concrete entities, is accompanied by the distinction between potential infinity for the domain of the mathematical and actual infinity, which affects factual reality (GM 4 93; GP 2 268-69; 282-283, 314). From this perspective, it is natural that the problem of the existence of infinitary objects moves to the field of what is actually real, since in the mathematical domain it is no longer a problem, since for Leibniz in that domain the infinite divisibility of the geometric continuum tends to weaken the intensity of the quarrel about the actual existence of infinitely small quantities, that is, smaller than any other quantity.[15] Instead of that, the actual infinite division of material bodies raises the need to seriously

[14] Apart from that, the notion of the infinitesimal that follows from the case of *Numeri infiniti* previously mentioned does not seem to coincide with the one we pointed out before: it is not a quantity smaller than any given one, but of a quantity smaller than any than can be given, or, as Leibniz literally says, a "last number".

[15] That is, they can be applied in mathematics as fictions without problems and can be substituted by other methods, but they do not exist in the actual world. An anonymous referee has objected that, in the question of Leibniz's treatment of infinitely small quantities, methodological and of existence questions must be distinguished, since Leibniz himself dealt with them separately. As an answer to this objection, we fully agree with this approach, as can be seen, for example, in a forthcoming paper of ours (Esquisabel y Raffo Quintana 2021). In the same way, our final brief reference to Leibniz's solution of the continuum problem, namely, the distinction between an ideal continuum (or "syncategorematic", in the sense of *potential*), and a real continuum, in which there is an infinite actual division, refers to the problem of existence, and not to methodological questions, regarding which Leibniz just appeals to infinitary fictions. On the other hand, it seems clear enough to us, as it is to Rabouin & Arthur (2020), that in his maturity Leibniz deals with the question of the justification for the introduction of infinitely small quantities by appealing to the principle of continuity.

deal with the question of the existence of infinitely small quantities, since the composition of the real continuum is at stake, as Leibniz notes in the quote of *Cum prodiisset*.

However, the introduction of infinitary notions from the purely mathematical point of view is not irrelevant either, since in one way or another it affects the justification of the effectiveness of the new calculus. Thus, the problem of impossibility as irrepresentability closely connects the question of fictionality with the principle of continuity, on the basis of which Leibniz tries to prove that infinitely small quantities are eliminable. According to this principle, when a series infinitely approaches a limiting case that does not belong to the series, it must be considered as included within the series (HOCD 40/Child 147; GM 5 385. Cf. Bos 1974, pp. 56-57). Thus, for example, if by doubling the sides of an inscribed polygon it is brought closer and closer to a circle, then the latter can be thought of as the last term in the series of polygons. Thus, we can conceive of the circle as an infinitangular polygon, although this is an irrepresentable fiction, except in an analogical way, by means of a finite polygon whose finite sides must be considered as if they were infinitesimal straight lines. Similarly, as in the examples in *Elementa nova matheseos universalis*, if a series of acute angles comes closer and closer to a right angle, the latter, which is the limit of the series, can also be thought of as an acute angle. Likewise, if a bundle of lines intersects another at further and further distances, thus moving the point of intersection further and further, it will happen that the limiting case of this bundle of lines will be a parallel line, which can be conceived as a straight line whose point of intersection is at infinity (A VI 4 521). In all these cases, it is a matter of conceiving irrepresentable or non-instantiable objects analogically in terms of representable geometric objects, thanks to which the possibilities of the analytical calculus are facilitated and expanded. Hence, it is not a matter of the existence or not of infinitary objects, but whether it is possible to give meaning to their introduction into the calculus or the geometric reasoning, as well as to the results obtained thanks to them, even if they are not able to be represented as such.[16]

Concluding remarks

Throughout our paper we have tried to show that, when Leibniz introduces mathematical fictions or fictional mathematical objects, he actually appeals to cognitively confused notions devoid of denotation, the use of which is validated, among other things, by efficiency in providing the resolution of mathematical problems. The question as to whether this resolution is also demonstrative or not remains open, because it implies examining with more precision what a mathematical proof for Leibniz consists of. Similarly, the question of the methods Leibniz uses to effectively introduce these kinds of confused notions into mathematics must wait for another work. In any case, mathematical fictions constitute a chapter of the Leibnizian concept of symbolic knowledge.

Be that as it may, the introduction of fictions into mathematics implies a problem in relation to mathematical objects in general, since Leibniz also gives them a purely "ideal" or "abstract" status, especially in his mature thinking. For this reason, we were forced to

[16] This role is often characterized in terms of introducing "ideal" concepts, as Sherry and Katz (2012) do. However, we think that the concept of "ideal", which corresponds to the concept introduced in the geometry of the nineteenth century, should be applied, in the case of Leibniz, *cum grano salis*. As we could show, towards the last stage of his thought, Leibniz conceives that all mathematical entities, and not only infinitary ones, are "ideal".

appeal to the difference between notion or concept, on the one hand and idea on the other, in order to show that fictions are notions "without idea" and, therefore, without denotation. The impossibility turned out to be precisely the criterion of the lack of denotation and, for that very reason, the mark of mathematical non-existence. Likewise, our examination showed that Leibniz holds three concepts of impossibility, giving rise to three concepts of fiction, namely, $fiction_1$ in terms of inconsistency, $fiction_2$ in terms of irrepresentability, and $fiction_3$ in terms of incompatibility with architectonic principles. The main goal of our analysis was to clarify the fictional status of infinitary concepts, that is, infinitesimal quantities and infinitely large quantities. According to our point of view, except for the concept of infinite number, which is inconsistent, the fictionality of infinitary quantities is for Leibniz based on considerations of irrepresentability and incompatibility with architectonic principles, that is, there is no way to represent or exhibit them geometrically and they cannot exist in the world as it was created. Although we have not analyzed it thoroughly, we have also argued that the impossibility of actual existence is for Leibniz more important than the impossibility based on irrepresentability, since the latter is admissible, insofar as it occurs in the field of mathematics and it arises from the methodological and analogical introduction of infinitary quantities.

However, the result of our inquiries raises some questions which remain unanswered. Firstly, the problem of the "ideality" of mathematical objects leads us to the following question: the fact that no geometric entity has an actual existence contaminates mathematics with a certain fictional look, in the sense that there is nothing in the real world that has the uniformity required by a geometric entity. Likewise, our conclusions confront us with a somewhat unforeseen result: infinitesimal quantities and infinitely large quantities, according to Leibniz, are not in themselves inconsistent, that is, they do not imply contradictions, even if unacceptable or paradoxical states of affairs follow from their *real* (non-mathematical) existence. If we are consistent, then it turns out that such objects could be possible in an absolute sense (although, as we saw, that possibility must be demonstrated) and, therefore, it could be argued that an idea corresponds to them and, what is more, we could conceive possible worlds in which they take place, even if they are "inharmonic" ones.[17] Probably, this consideration is in the background of the prudence of the mature Leibniz, when he expresses himself about the existence of infinitary objects, to the point of maintaining the presumption of impossibility, but not of affirming it categorically.

In conclusion, our final reflections point to a complex and current problem, even perhaps going beyond Leibniz, namely: what does it mean for Leibniz that a mathematical object exists? According to what we have suggested, mere consistency is a necessary condition, but apparently it is not enough, especially when we think about mathematics in terms of what constitutes something like the structural or "formal" background of reality, on the basis of which the mathematical science of nature is made possible.

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[17] One might wonder if other mathematical truths, different from those that rule our world, could be possible for Leibniz (as, for example, non-Euclidean geometries). The answer should apparently be negative, since mathematical truths are true in all possible worlds. But there are nuanced opinions on this topic. See, for example, Rescher 1981 and Debuiche and Rabouin 2019.

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